Semantic A-translations and Super-consistency entail Classical Cut Elimination

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Abstract. We show that if a theory R defined by a rewrite system is super-consistent, the classical sequent calculus modulo R enjoys the cut elimination property, which was an open question. For such theories it was already known that proofs strongly normalize in natural deduction modulo R, and that cut elimination holds in the intuitionistic sequent calculus modulo R.

We first define a syntactic and a semantic version of Friedman's A-translation, showing that it preserves the structure of pseudo-Heyting algebra, our semantic framework. Then we relate the interpretation of a theory in the A-translated algebra and its A-translation in the original algebra. This allows to show the stability of the super-consistency criterion and the cut elimination theorem.

Keywords: Deduction modulo, cut elimination, A-translation, pseudo-Heyting algebra, super-consistency.

1 Introduction

Deduction Modulo is a formalism that aims at separating computation from reasoning in proofs by making inferences *modulo* some congruence. This congruence is generated by rewrite rules on terms and on propositions, and, assuming confluence and termination, it is decidable by blind computation (normalization).

Rewrite rules on propositions is a key feature, allowing to express in a first-order setting *without* any axiom theories such as higher-order logic [8,10] or arithmetic [11]. Reasoning without axioms turns out to be a critical advantage for automated theorem provers [18,2,3,5] to not get lost during proof-search.

As a counterpart, fundamental properties such as cut elimination become a hard challenge. At the same time it is needed at both theoretical (consistency issues, e.g.) and practical levels, for instance to ensure the completeness of the proof-search algorithm of the aforementioned theorem provers. In the general case, it does not hold and this is why new techniques have been developed in order to ensure cut elimination for the widest possible range of rewrite systems.

Anticipating the definitions of Section 2, let us give two examples (see also Section 4.3) to illustrate the failure of cut elimination and/or normalization in general. For terminating (and confluent) examples, see [16]. The congruence generated by the rewrite system $P \to P \Rightarrow Q$ enables to prove the sequent $\vdash Q$ with a cut and this proof is neither normalizable in Natural deduction (the λ-term $(\lambda x.x\ x)\ (\lambda x.x\ x)$, that represents the aforementioned proof is typable) nor admits cut [10]. Instantiating Q by P yields the rewrite system $P \to P \Rightarrow P$. This allows for the same non-normalizing proof, while $\vdash A$ becomes provable in only two steps and without cut ; more generally, semantic means [16] show that in this case cut is admissible, showing the independence of normalization and cut elimination. All those questions are undecidable [6].

A first path to solve this problem, investigated in [10], is to show that a congruence has a reducibility candidate-valued model. Then any proof normalizes in natural deduction modulo this congruence. This propagates to cut elimination in intuitionistic sequent calculus modulo, but fails to directly extend to classical sequent calculus modulo. To fix this, a second derived criterion is proposed.

A second way is *super-consistency*, a notion developed in [7] that is a semantic criterion independent from reducibility candidates. It assumes the existence, for a given congruence, of a model for *any* pseudo-Heyting algebra. Since the reducibility candidates model of [10] is an instance of pseudo-Heyting algebra, this criterion implies that of [10], and all of its normalization / cut elimination corollaries. So this suffers the same drawback. A recent work [4] has also extended the criterion to the classical case, but still requires a modification of the criterion - specifically, pseudo-Heyting algebras become pre-Boolean algebras.

The beauty of super-consistency is that it is not hardwired for a particular deduction system. That is why it should indifferently prove cut elimination for the natural deduction, the intuitionistic as well as the classical sequent calculus. This is exactly what show here: cut-elimination for the classical sequent calculus modulo a given congruence, assuming the *unmodified* congruence has the *unmodified* super-consistency property.

After giving the definitions one would need to keep the paper as much self contained as possible, we introduce shortly the deduction modulo, relying on a basic knowledge first-order logic. Then we present the A-translation of propositions and rewrite systems [10], inspired by Friedman's A-translation [12], a refinement of double-negation translations, that bridges the intuitionistic and the classical worlds.

The core of the paper resides in the lifting of this translation on pseudo-Heyting algebras, at the semantic level. After verifying that all properties are preserved, we show that super-consistency is stable by A-translation: the rewrite system has a model in the translated algebra, so the translated rewrite system has a model in the original algebra.

Those results allow us to deduce that super-consistency is sufficient to prove cut-elimination in classical sequent calculus, propagating the normalization property of natural deduction modulo to cut elimination in intuitionistic and eventually classical sequent calculus, following [10].

2 Definitions

2.1 Pseudo-Heyting Algebra

Definition 1 (pseudo-Heyting algebra (pHA, [7])). *Let $\mathcal{B}$ be a set and $\leq$ a relation on it, $\mathcal{A}$ and $\mathcal{E}$ be subsets of $\wp(\mathcal{B})$, $\tilde{\top}$ and $\tilde{\bot}$ be elements of $\mathcal{B}$, $\tilde{\Rightarrow}$, $\tilde{\wedge}$, and $\tilde{\vee}$ be functions from $\mathcal{B} \times \mathcal{B}$ to $\mathcal{B}$, $\tilde{\forall}$ be a function from $\mathcal{A}$ to $\mathcal{B}$ and $\tilde{\exists}$ be a function from $\mathcal{E}$ to $\mathcal{B}$. The structure $\tilde{\mathcal{B}} = \langle \mathcal{B}, \leq, \mathcal{A}, \mathcal{E}, \tilde{\top}, \tilde{\bot}, \tilde{\Rightarrow}, \tilde{\wedge}, \tilde{\vee}, \tilde{\forall}, \tilde{\exists} \rangle$ is said to be a* pseudo-Heyting algebra *if for all a, b, c in $\mathcal{B}$, A in $\mathcal{A}$ and E in $\mathcal{E}$:*

1. *$a \leq a$ and if $a \leq b$, $b \leq c$ then $a \leq c$ ($\leq$ is a pre-order),*
2. *$a \leq \tilde{\top}$ and $\tilde{\bot} \leq a$ (maximum and minimum element),*
3. *$a \mathbin{\tilde{\wedge}} b \leq a$, $a \mathbin{\tilde{\wedge}} b \leq b$ and if $c \leq a$, $c \leq b$ then $c \leq a \mathbin{\tilde{\wedge}} b$,*
4. *$a \leq a \mathbin{\tilde{\vee}} b$, $b \leq a \mathbin{\tilde{\vee}} b$ and if $a \leq c$, $b \leq c$ then $a \mathbin{\tilde{\vee}} b \leq c$,*
5. *for any $x \in A$, $\tilde{\forall} A \leq x$ and if for any $x \in A$, $b \leq x$ then $b \leq \tilde{\forall} A$,*
6. *for any $x \in E$, $x \leq \tilde{\exists} E$ and if for any $x \in E$, $x \leq b$ then $\tilde{\exists} E \leq b$,*
7. *$a \leq b \mathbin{\tilde{\Rightarrow}} c$ iff $a \mathbin{\tilde{\wedge}} b \leq c$.*

Axioms for $\tilde{\wedge}$ and $\tilde{\forall}$ (resp. $\tilde{\vee}$ and $\tilde{\exists}$) confer them the property of a greatest lower bound (resp. lowest upper bound), while the unicity of the latters is not guaranteed, since $\leq$ is not anntisymmetric. Another guise of pHAs are Truth Value Algebras [7]. Also, $\tilde{\wedge}$ and $\tilde{\vee}$ are easily shown to be pre-commutative ($a \mathbin{\tilde{\wedge}} b \leq\geq b \mathbin{\tilde{\wedge}} a$) and pre-associative.

Definition 2 (Full [7]). *A pseudo-Heyting algebra is said to be* full *if $\mathcal{A} = \mathcal{E} = \wp(\mathcal{B})$,* i.e. *if $\tilde{\forall}\, A$ and $\tilde{\exists}\, A$ are defined for all $A \subset \mathcal{B}$.*

In this paper, all the pHA considered are full. When the pre-order is antisymmetric, then a full pHA is exactly a complete HA, in the terminology of [20]. In this paper, complete refers to the order $\sqsubseteq$ described below.

Definition 3 (Ordered pseudo-Heyting algebra). *A pseudo-Heyting algebra $\tilde{\mathcal{B}}$ is called* ordered *if it is equipped with an additional* order *relation $\sqsubseteq$ on $\mathcal{B}$ such that*

- *$\sqsubseteq$ is a refinement of $\leq$,* i.e. *if $a \sqsubseteq b$ then $a \leq b$,*
- *$\tilde{\top}$ is a maximal element,*
- *$\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\forall}$ and $\tilde{\exists}$ are monotonous, $\tilde{\Rightarrow}$ is left anti-monotonous and right monotonous.*

Definition 3 is an adapation to pHA of the corresponding definition of [7]. The "refinement condition" is shown in [7] to be a derived property (Proposition 4), but it is in fact trivially equivalent to the closure condition of $\tilde{\mathcal{B}}^+$.

Definition 4 (Complete ordered pseudo-Heyting algebra [7]). *An ordered pseudo-Heyting algebra $\tilde{\mathcal{B}}$ is said to be* complete *if every subset of $\mathcal{B}$ has a greatest lower bound for $\sqsubseteq$. Notice that this implies that every subset also has a least upper bound. We write $glb(a, b)$ and $lub(a, b)$ the greatest lower bound and the least upper bound of a and b for the order $\sqsubseteq$.*

The order relation $\sqsubseteq$ does not define a Heyting algebra order and, if by chance it does, the Heyting algebra operations may be different from those of $\tilde{\mathcal{B}}$.

2.2 Rewrite System

We work in usual predicate logic. Terms are variables and applied function symbols along their arity. Propositions are atoms (applied predicate symbols along their arity), and compound propositions with the help of connectives $\wedge, \vee, \Rightarrow, \top, \bot$ and quantifiers $\forall, \exists$. α-equivalent propositions are identified. To avoid parenthesis, $\Rightarrow$ and $\tilde{\Rightarrow}$ are considered to be *left* associative, therefore $A \Rightarrow B \Rightarrow B$ reads $(A \Rightarrow B) \Rightarrow B$. Negation is not a primitive connective, and is defined by $A \Rightarrow \bot$.

Definition 5 (Proposition rewrite rule). *We call* proposition rewrite rule *any rule $P \rightarrow A$ rewriting atomic propositions P into an arbitrary proposition A such that $\mathcal{FV}(A) \subseteq \mathcal{FV}(P)$.*

Definition 6 (Proposition rewrite system). *We define a* proposition rewrite system *as an orthogonal [19], hence confluent, set of proposition rewrite rules. The congruence generated by this rewrite system is noted $\equiv$.*

2.3 Interpretation

Definition 7 ($\tilde{\mathcal{B}}$-valued structure [7]). *Let $\mathcal{L} = \langle f_i, P_j \rangle$ be a language in predicate logic and $\tilde{\mathcal{B}}$ be a pHA, a $\tilde{\mathcal{B}}$-*valued structure *$\mathcal{M} = \langle \mathcal{M}, \tilde{\mathcal{B}}, \hat{f}_i, \hat{P}_j \rangle$ for the language $\mathcal{L}$ is a structure such that $\hat{f}_i$ is a function from $\mathcal{M}^n$ to $\mathcal{M}$ where n is the arity of the symbol f_i and $\hat{P}_j$ is a function from $\mathcal{M}^n$ to $\mathcal{B}$, the domain of $\tilde{\mathcal{B}}$, where n is the arity of the symbol P_i.*

Definition 8 (Denotation [7]). *Let $\tilde{\mathcal{B}}$ be a pHA, $\mathcal{M}$ be a $\tilde{\mathcal{B}}$-valued structure and ϕ be an assignment, i.e. a function associating elements of $\mathcal{M}$ to variables. The denotation in $\mathcal{M}$ of a proposition A or of a term t is defined as:*

- $[\![x]\!]_\phi = \phi(x)$,
- $[\![f(t_1, ..., t_n)]\!]_\phi = \hat{f}([\![t_1]\!]_\phi, ..., [\![t_n]\!]_\phi)$,
- $[\![P(t_1, ..., t_n)]\!]_\phi = \hat{P}([\![t_1]\!]_\phi, ..., [\![t_n]\!]_\phi)$,
- $[\![\top]\!]_\phi = \tilde{\top}$,
- $[\![\bot]\!]_\phi = \tilde{\bot}$,
- $[\![A \Rightarrow B]\!]_\phi = [\![A]\!]_\phi \ \tilde{\Rightarrow} \ [\![B]\!]_\phi$,
- $[\![A \wedge B]\!]_\phi = [\![A]\!]_\phi \ \tilde{\wedge} \ [\![B]\!]_\phi$,
- $[\![A \vee B]\!]_\phi = [\![A]\!]_\phi \ \tilde{\vee} \ [\![B]\!]_\phi$,
- $[\![\forall x \ A]\!]_\phi = \tilde{\forall} \ \{[\![A]\!]_{\phi+\langle x,e \rangle} \mid e \in \mathcal{M}\}$,
- $[\![\exists x \ A]\!]_\phi = \tilde{\exists} \ \{[\![A]\!]_{\phi+\langle x,e \rangle} \mid e \in \mathcal{M}\}$.

The denotation of a proposition containing quantifiers is always defined if the pHA is full, otherwise it may be undefined.

Definition 9 (Model [7]). *The $\tilde{\mathcal{B}}$-valued structure $\mathcal{M}$ is said to be* a model of *a rewrite system R if for any two propositions A, B such that $A \equiv B$, $[\![A]\!] = [\![B]\!]$.*

Soundness and completeness hold [7]: the sequent $\Gamma \vdash B$ is provable if and only if $[\![\Gamma]\!] \leq [\![B]\!]$ for any pseudo-Heyting algebra $\tilde{\mathcal{B}}$ and any model interpretation for R in $\tilde{\mathcal{B}}$. The direct way is an usual induction [7], while the converse is a direct consequence of the completeness theorem with respect to Heyting algebra. For instance one can construct the Lindenbaum algebra [7], or a context-based algebra [17].

2.4 Classical Sequent Calculus Modulo

Figure 1 recalls the classical sequent calculus modulo. It depends on a congruence $\equiv$ determined by a fixed rewrite system R. If R is empty $\equiv$ boils down to syntactic equality and we get usual sequent calculus. The intuitionistic sequent calculus modulo has the same rules, except that the right-hand sides of sequents contain at most *one* proposition. Two rules are impacted: $\vee$-r splits into two rules $\vee_1$ and $\vee_2$, and, in the right premiss of the $\Rightarrow$-left rule, Δ is overwritten by A.

identity group

$$\text{axiom, } A \equiv B \; \frac{}{A \vdash B} \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma \vdash \Delta} \; \text{cut, } A \equiv B$$

logical group

$$\wedge\text{-l, } C \equiv A \wedge B \; \frac{\Gamma, A, B \vdash \Delta}{\Gamma, C \vdash \Delta} \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash C, \Delta} \; \wedge\text{-r, } C \equiv A \wedge B$$

$$\vee\text{-l, } C \equiv A \vee B \; \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, C \vdash \Delta} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash C, \Delta} \; \vee\text{-r, } C \equiv A \vee B$$

$$\Rightarrow\text{-l, } C \equiv A \Rightarrow B \; \frac{\Gamma, B \vdash \Delta \qquad \Gamma \vdash A, \Delta}{\Gamma, C \vdash \Delta} \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash C, \Delta} \; \Rightarrow\text{-r, } C \equiv A \Rightarrow B$$

$$\bot\text{-l, } A \equiv \bot \; \frac{}{A \vdash} \qquad \frac{}{\vdash A} \; \top\text{-r, } A \equiv \top$$

$$\forall\text{-l, } B \equiv \forall x A \; \frac{\Gamma, \{t/x\}A \vdash \Delta}{\Gamma, B \vdash \Delta} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash B, \Delta} \; \forall\text{-r, } B \equiv \forall x A, \; x \text{ fresh}$$

$$\exists\text{-l, } B \equiv \exists x A, \; x \text{ fresh} \; \frac{\Gamma, A \vdash \Delta}{\Gamma, B \vdash \Delta} \qquad \frac{\Gamma \vdash \{t/x\}A, \Delta}{\Gamma \vdash B, \Delta} \; \exists\text{-r, } B \equiv \exists x A$$

structural group

$$\text{contr-l, } A \equiv B_1 \equiv B_2 \; \frac{\Gamma, B_1, B_2 \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash B_1, B_2, \Delta}{\Gamma \vdash A, \Delta} \; \text{contr-r, } A \equiv B_1 \equiv B_2$$

$$\text{weak-l} \; \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \; \text{weak-r}$$

Fig. 1. Classical sequent calculus modulo

2.5 Super-consistency

Definition 10 (Super-consistency [7]). *A rewrite system R (a congruence $\equiv$) in deduction modulo is* super-consistent *if it has a $\tilde{\mathcal{B}}$-valued model for all full, ordered and complete pseudo-Heyting algebra $\tilde{\mathcal{B}}$.*

Super-consistency is akin to consistency with respect to *all* pHA. Note that the choice of the structure (Definition 7) is open. Considering only HA is not enough, as the rewrite system $P \to P \Rightarrow P$ devised in Section 1 , as well as the one of Section 4.3 would then be super-consistent but not normalizing.

3 A-translations

Instead of first performing a negative translation [10] and then the proper A-translation, as in the original work of Friedman [12], we consider a variant of the composition of both.

3.1 Syntactic Translation of a Proposition

Definition 11 (A-translation of a proposition).
Let B be a proposition. Let A be a proposition in which free variables are not bound by quantifiers in B. A is said B-unbound. We let B^A be:

- $B^A = B$ *if* B *is atomic,*
- $\top^A = \top$,
- $\bot^A = \bot$,
- $(B \Rightarrow C)^A = (B^A \Rightarrow A \Rightarrow A) \Rightarrow (C^A \Rightarrow A \Rightarrow A)$,
- $(B \wedge C)^A = (B^A \Rightarrow A \Rightarrow A) \wedge (C^A \Rightarrow A \Rightarrow A)$,
- $(B \vee C)^A = (B^A \Rightarrow A \Rightarrow A) \vee (C^A \Rightarrow A \Rightarrow A)$,
- $(\forall x\ B)^A = \forall x\ (B^A \Rightarrow A \Rightarrow A)$,
- $(\exists x\ B)^A = \exists x\ (B^A \Rightarrow A \Rightarrow A)$.

Remark 1. Kolmogorov's double negation translation [10] of B is $\neg\neg B^{\bot}$. As well as this translation has been simplified by Gödel, Gentzen and others [14,13,20], we can also simplify Definition 11 so that it introduces less A.

Definition 12 (A-translation of a rewrite system). *Let $R = \{P_i \to A_i\}$ be a proposition rewrite system and A be a formula that is A_i-unbound for all i. We define its A-translation, written R^A, as $\{P_i \to A_i^A\}$.*

3.2 Semantic a-translation of a pHA

We now lift the A-translation process at the semantic level.

Definition 13 (Semantic a-translation). *Let $\tilde{\mathcal{B}}$ be the full pseudo-Heyting algebra $\langle \mathcal{B}, \leq, \wp(\mathcal{B}), \wp(\mathcal{B}), \tilde{\top}, \tilde{\bot}, \tilde{\Rightarrow}, \tilde{\wedge}, \tilde{\vee}, \tilde{\forall}, \tilde{\exists} \rangle$ and let $a \in \mathcal{B}$.*

We let $\tilde{\mathcal{B}}^a$ be the structure $\langle \mathcal{B}, \overset{a}{\leq}, \wp(\mathcal{B}), \wp(\mathcal{B}), \overset{a}{\top}, \overset{a}{\bot}, \overset{a}{\Rightarrow}, \overset{a}{\wedge}, \overset{a}{\vee}, \overset{a}{\forall}, \overset{a}{\exists} \rangle$, that we call the a-translation of $\tilde{\mathcal{B}}$, where:

- $b \overset{a}{\leq} c$ *iff* $b \tilde{\Rightarrow} a \tilde{\Rightarrow} a \leq c \tilde{\Rightarrow} a \tilde{\Rightarrow} a$,
- $\overset{a}{\top} \triangleq \tilde{\top}$,
- $\overset{a}{\bot} \triangleq \tilde{\bot}$,
- $b \overset{a}{\Rightarrow} c \triangleq ((b \tilde{\Rightarrow} a \tilde{\Rightarrow} a) \tilde{\Rightarrow} (c \tilde{\Rightarrow} a \tilde{\Rightarrow} a))$,
- $b \overset{a}{\wedge} c \triangleq ((b \tilde{\Rightarrow} a \tilde{\Rightarrow} a) \tilde{\wedge} (c \tilde{\Rightarrow} a \tilde{\Rightarrow} a))$,
- $b \overset{a}{\vee} c \triangleq ((b \tilde{\Rightarrow} a \tilde{\Rightarrow} a) \tilde{\vee} (c \tilde{\Rightarrow} a \tilde{\Rightarrow} a))$,
- $\overset{a}{\forall} A \triangleq (\tilde{\forall}\ (A \tilde{\Rightarrow} a \tilde{\Rightarrow} a))$,
- $\overset{a}{\exists} A \triangleq (\tilde{\exists}\ (A \tilde{\Rightarrow} a \tilde{\Rightarrow} a))$.

with the convention that, for any $A \subseteq \mathcal{B}$, $A \tilde{\Rightarrow} a \tilde{\Rightarrow} a = \{b \tilde{\Rightarrow} a \tilde{\Rightarrow} a \mid b \in A\}$.

We may straightforwardly check that $\langle \mathcal{B}, \overset{a}{\leq}, \wp(\mathcal{B}), \wp(\mathcal{B}), \overset{a}{\top}, \overset{a}{\bot}, \overset{a}{\Rightarrow}, \overset{a}{\wedge}, \overset{a}{\vee}, \overset{a}{\forall}, \overset{a}{\exists} \rangle$ is a valid *structure*, in the sense that $\overset{a}{\leq}$ operators are well-defined; in particular $\overset{a}{\forall}$ and $\overset{a}{\exists}$ are defined for any subset of $\mathcal{B}$. We show below that it is also a full, ordered and complete pHA.

4 Results

4.1 On the a-translation of a pHA

We recall some useful facts about the semantic implication that hold in pseudo-Heyting algebras:

Proposition 1. *Let* $\mathcal{B}$ *be a pHA and* $a, b, c \in \tilde{\mathcal{B}}$ *such that* $b \leq c$*. Then:*

$$b \leq a \tilde{\Rightarrow} b \tag{1}$$
$$a \tilde{\Rightarrow} b \mathbin{\tilde{\wedge}} a \leq b \tag{2}$$
$$b \leq b \tilde{\Rightarrow} a \tilde{\Rightarrow} a \tag{3}$$
$$a \tilde{\Rightarrow} b \leq a \tilde{\Rightarrow} c \tag{4}$$
$$c \tilde{\Rightarrow} a \leq b \tilde{\Rightarrow} a \tag{5}$$
$$b \tilde{\Rightarrow} a \tilde{\Rightarrow} a \leq c \tilde{\Rightarrow} a \tilde{\Rightarrow} a \tag{6}$$
$$b \tilde{\Rightarrow} a \tilde{\Rightarrow} a \tilde{\Rightarrow} a \leq b \tilde{\Rightarrow} a \tag{7}$$

Proof. Standard, using the definition of $\tilde{\Rightarrow}$. Let us show 7: by 3 $b \leq b \tilde{\Rightarrow} a \tilde{\Rightarrow} a \leq b \tilde{\Rightarrow} a \tilde{\Rightarrow} a \tilde{\Rightarrow} a \tilde{\Rightarrow} a$. Then by definition of $\tilde{\Rightarrow}$ we get first $(b \tilde{\Rightarrow} a \tilde{\Rightarrow} a \tilde{\Rightarrow} a) \mathbin{\tilde{\wedge}} b \leq a$ and then $b \tilde{\Rightarrow} a \tilde{\Rightarrow} a \tilde{\Rightarrow} a \leq b \tilde{\Rightarrow} a$. □

Proposition 2. *If* $\tilde{\mathcal{B}}$ *is a full pHA then its* a*-translation* $\overset{a}{\mathcal{B}}$ *is a full pHA.*

Proof. We check one by one all the points of Definition 1 and Definition 2:

- $\overset{a}{\leq}$ is a pre-order: inherited from $\leq$
- $b \overset{a}{\leq} \overset{a}{\top}$ since $b \tilde{\Rightarrow} a \tilde{\Rightarrow} a \leq \tilde{\top} \tilde{\Rightarrow} a \tilde{\Rightarrow} a$ (by 6). Similarly for $\overset{a}{\bot}$.
- $b \overset{a}{\wedge} c$ is a lower bound of b and c. Let us show $b \overset{a}{\wedge} c \overset{a}{\leq} b$. By definition of $\overset{a}{\wedge}$ and of $\tilde{\wedge}$, $b \overset{a}{\wedge} c \leq b \tilde{\Rightarrow} a \tilde{\Rightarrow} a$. By 6 $(b \overset{a}{\wedge} c) \tilde{\Rightarrow} a \tilde{\Rightarrow} a \leq b \tilde{\Rightarrow} a \tilde{\Rightarrow} a \tilde{\Rightarrow} a \tilde{\Rightarrow} a$ and by 7 of Proposition 1 $b \tilde{\Rightarrow} a \tilde{\Rightarrow} a \tilde{\Rightarrow} a \tilde{\Rightarrow} a \leq b \tilde{\Rightarrow} a \tilde{\Rightarrow} a$ which allows us to conclude. Similar arguments show that $b \overset{a}{\wedge} c \overset{a}{\leq} c$.

- $b \overset{a}{\wedge} c$ is a greatest lower bound of b and c: let d such that $d \overset{a}{\leq} b$ and $d \overset{a}{\leq} c$. By definition of $\tilde{\wedge}$, $\overset{a}{\wedge}$ and of $\overset{a}{\leq}$, $d \Rrightarrow a \Rrightarrow a \leq b \overset{a}{\wedge} c$ and by 3 of Proposition 1, $b \overset{a}{\wedge} c \leq (b \overset{a}{\wedge} c) \Rrightarrow a \Rrightarrow a$ which allows us to conclude.
- $b \overset{a}{\vee} c$ is an upper bound of b and c. Let us show $b \overset{a}{\leq} b \overset{a}{\vee} c$. By definition of $\tilde{\vee}$ and of $\overset{a}{\vee}$, $b \Rrightarrow a \Rrightarrow a \leq b \overset{a}{\vee} c$. We conclude by 3 of Proposition 1. Similar arguments show that $c \overset{a}{\leq} b \overset{a}{\vee} c$.
- $b \overset{a}{\vee} c$ is a least upper bound of b and c. Let d such that $b \overset{a}{\leq} d$ and $c \overset{a}{\leq} d$. Then, $(b \Rrightarrow a \Rrightarrow a) \tilde{\vee} (c \Rrightarrow a \Rrightarrow a) \leq d \Rrightarrow a \Rrightarrow a$ and by 6 of Proposition 1, $((b \Rrightarrow a \Rrightarrow a) \tilde{\vee} (c \Rrightarrow a \Rrightarrow a)) \Rrightarrow a \Rrightarrow a \leq d \Rrightarrow a \Rrightarrow a \Rrightarrow a \Rrightarrow a$. By applying 7, $d \Rrightarrow a \Rrightarrow a \Rrightarrow a \Rrightarrow a \leq d \Rrightarrow a \Rrightarrow a$, which allows us to conclude.
- $\overset{a}{\forall} A$ is a lower bound of A. Let $x \in A$. Then $\overset{a}{\forall} A \leq x \Rrightarrow a \Rrightarrow a$ by definition of $\overset{a}{\forall}$ and $\tilde{\forall}$. Using Proposition 1, by 6 $(\overset{a}{\forall} A) \Rrightarrow a \Rrightarrow a \leq x \Rrightarrow a \Rrightarrow a \Rrightarrow a \Rrightarrow a$ and by 7 $x \Rrightarrow a \Rrightarrow a \Rrightarrow a \Rrightarrow a \leq x \Rrightarrow a \Rrightarrow a$, which allows us to conclude.
- $\overset{a}{\forall} A$ is a greatest lower bound of A. Let b such that for any $x \in A$, $b \overset{a}{\leq} x$. Then $b \Rrightarrow a \Rrightarrow a \leq x \Rrightarrow a \Rrightarrow a$ and by definition of $\tilde{\forall}$, $b \Rrightarrow a \Rrightarrow a \leq \tilde{\forall}(A \Rrightarrow a \Rrightarrow a) = \overset{a}{\forall} A$. By 3 of Proposition 1, $\overset{a}{\forall} A \leq (\overset{a}{\forall} A) \Rrightarrow a \Rrightarrow a$, which allows us to conclude.
- $\overset{a}{\exists} A$ is an upper bound of A. Let $x \in A$. Then $x \Rrightarrow a \Rrightarrow a \leq \overset{a}{\exists} A$ by definition of $\overset{a}{\exists}$ and $\tilde{\exists}$. By 3 $\overset{a}{\exists} A \leq (\overset{a}{\exists} A) \Rrightarrow a \Rrightarrow a$, which allows us to conclude.
- $\overset{a}{\exists} A$ is a least upper bound of A. Let b such that for any $x \in A$, $x \overset{a}{\leq} b$. Then $x \Rrightarrow a \Rrightarrow a \leq b \Rrightarrow a \Rrightarrow a$ and by definition of $\tilde{\exists}$, $\overset{a}{\exists} A = \tilde{\exists}(A \Rrightarrow a \Rrightarrow a) \leq b \Rrightarrow a \Rrightarrow a$. By Proposition 1 we derive $(\overset{a}{\exists} A) \Rrightarrow a \Rrightarrow a \leq b \Rrightarrow a \Rrightarrow a \Rrightarrow a \Rrightarrow a$ and $b \Rrightarrow a \Rrightarrow a \Rrightarrow a \Rrightarrow a \leq b \Rrightarrow a \Rrightarrow a$, which allows us to conclude.
- direct way of the implication property. Assume $b \overset{a}{\leq} c \overset{a}{\Rrightarrow} d$, that is to say $b \Rrightarrow a \Rrightarrow a \leq ((c \Rrightarrow a \Rrightarrow a) \Rrightarrow (d \Rrightarrow a \Rrightarrow a)) \Rrightarrow a \Rrightarrow a$. As an intermediate result we claim that for any x, y and z, $(x \Rrightarrow (y \Rrightarrow z)) \Rrightarrow z \Rrightarrow a \leq x \Rrightarrow (y \Rrightarrow a)$.

$$\begin{array}{rll}
x \Rrightarrow (y \Rrightarrow z) \leq & x \Rrightarrow (y \Rrightarrow z) & \text{(reflexivity)} \\
(x \Rrightarrow (y \Rrightarrow z)) \tilde{\wedge} x \tilde{\wedge} y \leq & z & \text{(Definition of } \Rrightarrow\text{)} \\
x \tilde{\wedge} y \leq & x \Rrightarrow (y \Rrightarrow z) \Rrightarrow z & \text{(Definition of } \Rrightarrow\text{)} \\
x \tilde{\wedge} y \leq & [x \Rrightarrow (y \Rrightarrow z) \Rrightarrow z] \Rrightarrow a \Rrightarrow a & \text{(Proposition 1)} \\
[x \Rrightarrow (y \Rrightarrow z) \Rrightarrow z \Rrightarrow a] \tilde{\wedge} x \tilde{\wedge} y \leq & a & \text{(Definition of } \Rrightarrow\text{)} \\
x \Rrightarrow (y \Rrightarrow z) \Rrightarrow z \Rrightarrow a \leq & x \Rrightarrow (y \Rrightarrow a) & \text{(Definition of } \Rrightarrow\text{)}
\end{array}$$

  If we replace in this last inequality x by $c \Rrightarrow a \Rrightarrow a$, y by $d \Rrightarrow a$ and z by a, we get $((c \Rrightarrow a \Rrightarrow a) \Rrightarrow (d \Rrightarrow a \Rrightarrow a)) \Rrightarrow a \Rrightarrow a \leq ((c \Rrightarrow a \Rrightarrow a) \Rrightarrow (d \Rrightarrow a \Rrightarrow a))$ so that we derive $b \Rrightarrow a \Rrightarrow a \leq (c \Rrightarrow a \Rrightarrow a) \Rrightarrow (d \Rrightarrow a \Rrightarrow a)$, or said otherwise $(b \Rrightarrow a \Rrightarrow a) \tilde{\wedge} (c \Rrightarrow a \Rrightarrow a) \leq d \Rrightarrow a \Rrightarrow a$. By Proposition 1 we get the inequality $((b \Rrightarrow a \Rrightarrow a) \tilde{\wedge} (c \Rrightarrow a \Rrightarrow a)) \Rrightarrow a \Rrightarrow a \leq d \Rrightarrow a \Rrightarrow a \Rrightarrow a \Rrightarrow a \leq d \Rrightarrow a \Rrightarrow a$, which is exactly $b \overset{a}{\wedge} c \overset{a}{\leq} d$.
- conversely, assume $b \overset{a}{\wedge} c \overset{a}{\leq} d$, i.e. $((b \Rrightarrow a \Rrightarrow a) \tilde{\wedge} (c \Rrightarrow a \Rrightarrow a)) \Rrightarrow a \Rrightarrow a \leq d \Rrightarrow a \Rrightarrow a$. By 6 of Proposition 1 we get that $((b \Rrightarrow a \Rrightarrow a) \tilde{\wedge} (c \Rrightarrow a \Rrightarrow a)) \leq ((b \Rrightarrow a \Rrightarrow a) \tilde{\wedge} (c \Rrightarrow a \Rrightarrow a)) \Rrightarrow a \Rrightarrow a$, so $b \Rrightarrow a \Rrightarrow a \leq (c \Rrightarrow a \Rrightarrow a) \Rrightarrow (d \Rrightarrow a \Rrightarrow a)$ by definition of $\Rrightarrow$. And by 3 we get that $(c \Rrightarrow a \Rrightarrow a) \Rrightarrow (d \Rrightarrow a \Rrightarrow a) \leq ((c \Rrightarrow a \Rrightarrow a) \Rrightarrow (d \Rrightarrow a \Rrightarrow a)) \Rrightarrow a \Rrightarrow a$, which allows us to conclude. □

Proposition 3. *Let $\mathcal{B}$ be a full and ordered pHA, with respect to $\sqsubseteq$. Let $a \in \tilde{\mathcal{B}}$. The a-translation $\mathcal{B}^a$ of $\mathcal{B}$ is a full and ordered pHA with respect to $\sqsubseteq$.*

Proof. By Proposition 2, $\mathcal{B}^a$ is a full pHA. We check Definition 3:

- $\sqsubseteq$ is by definition an order relation on $\mathcal{B}$, which is also the domain of $\tilde{\mathcal{B}}^a$.
- $\overset{a}{\top}$ (resp. $\overset{a}{\bot}$) is maximal (resp. minimal) for the same reason.
- assume $b \sqsubseteq c$. Then $b \leq c$ and by Proposition 1 $b \overset{a}{\leq} c$.
- $\overset{a}{\wedge}$ is monotonous. Let b, c, d be elements of the algebra, and assume $b \sqsubseteq c$. By left-antimonotonousity of $\sqsubseteq$ with respect to $\tilde{\Rightarrow}$, $b \tilde{\Rightarrow} a \tilde{\Rightarrow} a \sqsubseteq c \tilde{\Rightarrow} a \tilde{\Rightarrow} a$, so $b \overset{a}{\wedge} d = (b \tilde{\Rightarrow} a \tilde{\Rightarrow} a) \tilde{\wedge} (d \tilde{\Rightarrow} a \tilde{\Rightarrow} a) \sqsubseteq (c \tilde{\Rightarrow} a \tilde{\Rightarrow} a) \tilde{\wedge} (d \tilde{\Rightarrow} a \tilde{\Rightarrow} a) = c \overset{a}{\wedge} d$ by monotonicity of $\sqsubseteq$ with respect to $\tilde{\wedge}$.
- the other properties with respect to $\overset{a}{\vee}$, $\overset{a}{\Rightarrow}$, $\overset{a}{\forall}$ and $\overset{a}{\exists}$ are shown in the same way: first notice that $b \tilde{\Rightarrow} a \tilde{\Rightarrow} a \sqsubseteq c \tilde{\Rightarrow} a \tilde{\Rightarrow} a$ and then use the corresponding property of $\sqsubseteq$ with respect to the original connective. Remember that, for A, A' sets of elements of $\tilde{\mathcal{B}}^a$, $A \sqsubseteq A'$ means that, for any $x \in A$, there exists $y \in A'$ such that $x \sqsubseteq y$. □

Proposition 4. *If $\mathcal{B}$ is a full, ordered and complete pHA, then its a-translation $\mathcal{B}^a$ is a full, ordered and complete pHA.*

Proof. From Proposition 3, $\tilde{\mathcal{B}}^a$ is full and ordered. The greatest lower and lowest upper bounds of any A subset of $\mathcal{B}$ (the domain of $\tilde{\mathcal{B}}^a$) for $\sqsubseteq$ are members of $\mathcal{B}$ because $\tilde{\mathcal{B}}$ is complete. The condition of Definition 4 is fulfilled. □

4.2 Relating Interpretations

Proposition 5. *Let $\mathcal{B}$ be a full, ordered and complete pHA. Consider a $\tilde{\mathcal{B}}$-valued structure $\mathcal{M}$ and note $[\![.]\!]$ the denotation $\mathcal{M}$ generates in $\tilde{\mathcal{B}}$. Let A be a closed proposition and let $B^{[\![A]\!]}$ be the $[\![A]\!]$-translation of $\mathcal{B}$:*

1. *$\mathcal{M}$ is also a $\tilde{\mathcal{B}}^{[\![A]\!]}$-valued structure. Let $[\![.]\!]^{[\![A]\!]}$ be the denotation it generates in $\tilde{\mathcal{B}}^{[\![A]\!]}$.*
2. *for any term t, any assignment ϕ, $[\![t]\!]_\phi = [\![t]\!]_\phi^{[\![A]\!]}$.*
3. *For any proposition B, any assignment ϕ, $[\![B^A]\!]_\phi = [\![B]\!]_\phi^{[\![A]\!]}$.*

A is chosen to be closed, otherwise we would need to consider $[\![A]\!]_{\phi_0}$ for a fixed ϕ_0 and consider only formulæ B such that A is B-unbound. We rather avoid those complications.

Proof. $\mathcal{M}$ is obviously a $\tilde{\mathcal{B}}^{[\![A]\!]}$-valued structure (see Definition 7) since the domain of both pHAs is the same and $\mathcal{M}$ assigns values only to atomic constructs. The second claim is also obvious, since the domain for terms does not change. We prove the last claim by an easy induction on the structure of B, where we omit the valuation ϕ, which plays no role. We note $a = [\![A]\!]$ in the definition of the operators of $\tilde{\mathcal{B}}^{[\![A]\!]}$.

– if B is an atomic formula $P(t_1, \cdots, t_n)$, then by construction and definition of the A-translation:

$$\llbracket B^A \rrbracket = \llbracket B \rrbracket = \hat{P}(\llbracket t_1 \rrbracket, \cdots, \llbracket t_n \rrbracket) = \hat{P}(\llbracket t_1 \rrbracket^{\llbracket A \rrbracket}, \cdots, \llbracket t_n \rrbracket^{\llbracket A \rrbracket}) = \llbracket B \rrbracket^{\llbracket A \rrbracket}$$

– $\llbracket \top^A \rrbracket = \tilde{\top} = \llbracket \top \rrbracket^{\llbracket A \rrbracket}$, similarly for $\bot$.

– $\llbracket (B \Rightarrow C)^A \rrbracket = (\llbracket B^A \rrbracket \tilde{\Rightarrow} \llbracket A \rrbracket \tilde{\Rightarrow} \llbracket A \rrbracket) \tilde{\Rightarrow} (\llbracket C^A \rrbracket \tilde{\Rightarrow} \llbracket A \rrbracket \tilde{\Rightarrow} \llbracket A \rrbracket) = \llbracket B^A \rrbracket \overset{a}{\tilde{\Rightarrow}} \llbracket C^A \rrbracket$ which, by induction hypothesis is equal to $\llbracket B \rrbracket^{\llbracket A \rrbracket} \overset{a}{\tilde{\Rightarrow}} \llbracket C \rrbracket^{\llbracket A \rrbracket} = \llbracket B \Rightarrow C \rrbracket^{\llbracket A \rrbracket}$.

– similarily for $\wedge$ and $\vee$.

– $\llbracket \forall x B^A \rrbracket = \tilde{\forall}\{\llbracket B^A \rrbracket_{\langle x,d \rangle} \tilde{\Rightarrow} \llbracket A \rrbracket \tilde{\Rightarrow} \llbracket A \rrbracket \mid d \in \mathcal{M}\}$ and by induction hypothesis and the notation of Definition 13, this is equal to $\tilde{\forall}\{\llbracket B \rrbracket^{\llbracket A \rrbracket}_{\langle x,d \rangle} \mid d \in \mathcal{M}\} \tilde{\Rightarrow} \llbracket A \rrbracket \tilde{\Rightarrow} \llbracket A \rrbracket = \overset{a}{\tilde{\forall}}\{\llbracket B \rrbracket^{\llbracket A \rrbracket}_{\langle x,d \rangle} \mid d \in \mathcal{M}\} = \llbracket \forall x B \rrbracket^{\llbracket A \rrbracket}$.

– similarly for $\exists$. □

4.3 Stability of Super-consistency

In this section we show that the super-consistency property of a rewrite system is preserved by A-translation under certain conditions.

First, notice that the general statement is not true because nasty interferences can happen if the A-translation is done with respect to a A containing propositions of the rewrite system. In particular, we can lose the normalization property, which is implied by super-consistency, and so, super-consistency itself. To illustrate this, consider the following rewrite system consisting of the sole rule $P \to \top \wedge \top$. Super-consistency comes out easily: given a pHA $\tilde{\mathcal{B}}$, we let $\hat{P} = \tilde{\top} \tilde{\wedge} \tilde{\top}$. But super-consistency fails for its P-translated rewrite system:

$$P \to (\top \Rightarrow P \Rightarrow P) \wedge (\top \Rightarrow P \Rightarrow P)$$

$$\frac{\Gamma \vdash \pi_1 : A \qquad \Gamma \vdash \pi_2 : B}{\Gamma \vdash \langle \pi_1, \pi_2 \rangle : C} \wedge_i, A \wedge B \equiv C \qquad \frac{\Gamma, x : A \vdash \pi : B}{\Gamma \vdash \lambda x.\pi : C} \Rightarrow_i, C \equiv A \Rightarrow B$$

$$\frac{\Gamma \vdash \pi : C}{\Gamma \vdash fst(\pi) : A} \wedge_{e1}, C \equiv A \wedge B \qquad \frac{\Gamma \vdash \pi_1 : C \qquad \Gamma \vdash \pi_2 : A}{\Gamma \vdash \pi_1 \; \pi_2 : B} \Rightarrow_e, C \equiv A \Rightarrow B$$

$$fst\langle \pi_1, \pi_2 \rangle \rhd \pi_1 \qquad (\lambda x.\pi_1) \; \pi_2 \rhd \{\pi_2/x\}\pi_1$$

Fig. 2. Some typing and reduction rules of natural deduction modulo [10]

As we will see, in natural deduction we can define a proof-term that is not normalizing. Adopting the syntax and typing rules of [10], shown in Figure 2, we let t_1 and t_2 be the following λ-terms, I being the constant corresponding to the $\top$-intro rule:[3]

$$t_1 = \lambda x.[fst(x\ I)\ (\lambda z.(x\ I))]$$
$$t_2 = \lambda z.\langle t_1, t_1\rangle$$

Those terms can be typed respectively by $\top \Rightarrow P \Rightarrow P$ and by $\top \Rightarrow (\top \Rightarrow P \Rightarrow P \wedge \top \Rightarrow P \Rightarrow P)$ or, using the congruence, by $\top \Rightarrow P$: both bound z can be assigned the type $\top$, while x has the type $\top \Rightarrow P \equiv \top \Rightarrow (\langle(\top \Rightarrow P) \Rightarrow P), (\top \Rightarrow P) \Rightarrow P)\rangle$, this last type identification being the source of the problems. With those terms, we form the following looping reduction sequence:

$$\begin{aligned} t_1\ t_2 &\rhd fst(t_2\ I)\ (\lambda z.(t_2\ I)) \\ &\rhd fst(\langle t_1, t_1\rangle)\ (\lambda z.\langle t_1, t_1\rangle) \\ &\rhd t_1\ t_2 \end{aligned}$$

Since we do not have normalization, we cannot have super-consistency. This is why restricting A is the key to Theorem 1.

Definition 14 (R-compatibility). *Let R be a rewriting system. A proposition A is said to be R-compatible if and only if does not contain any predicate or function symbol appearing in R.*

Proposition 6. *Let R be a rewrite system, and A be a closed proposition. Let $\tilde{\mathcal{B}}$ be a pHA and consider a $\tilde{\mathcal{B}}$-valued structure $\mathcal{M}$, generating an interpretation $[\![\_]\!]$. Let $\tilde{\mathcal{B}}^{[\![A]\!]}$ be the $[\![A]\!]$-translation of $\tilde{\mathcal{B}}$ and R^A be the A-translation of R.*

If the interpretation $[\![\_]\!]^{[\![A]\!]}$ generated by $\mathcal{M}$ in $\tilde{\mathcal{B}}^{[\![A]\!]}$ is a model of R then R^A has a $\mathcal{B}$-model.

Proof. Let $P \rightarrow F^A \in R^A$. By hypothesis, $P \rightarrow F \in R$ and $[\![P]\!]^{[\![A]\!]} = [\![F]\!]^{[\![A]\!]}$. We conclude by noticing that, by definition, $[\![P]\!]^{[\![A]\!]} = [\![P]\!]$ and that, by Proposition 5, $[\![F]\!]^{[\![A]\!]} = [\![F^A]\!]$. □

The main requirement of Proposition 6 is that $[\![\_]\!]^{[\![A]\!]}$ must be a model of R. The choice of $[\![\_]\!]$ is here a degree of freedom, but this is not sufficient, even assuming super-consistency. Indeed, the example of the beginning of the section shows that this is impossible if A is not R-compatible. We must go through the following definition lemma.

Lemma 1 (Relative grafting of structures). *Let $\tilde{\mathcal{B}}$ be a pHA and $\mathcal{M}_0$ and $\mathcal{M}_1$ be two $\tilde{\mathcal{B}}$-valued structures. Let A be a proposition. We define $\mathcal{M}_2$, the A-grafting of $\mathcal{M}_0$ onto $\mathcal{M}_1$ as the following $\tilde{\mathcal{B}}$-structure:*

- *for any function symbol f, $\hat{f} = \hat{f}_0$ (the value assigned by $\mathcal{M}_0$) if f syntactically appears in A and $\hat{f} = \hat{f}_1$ (the value assigned by $\mathcal{M}_1$) otherwise.*

[3] At the price of readability, I and $\top$ can be everywhere safely replaced by $\lambda y.y$ and $B \Rightarrow B$, respectively.

– *for any predicate symbol* P, $\hat{P} = \hat{P}_0$ *(the value assigned by* $\mathcal{M}_0$*) if* P *syntactically appears in* A *and* $\hat{P} = \hat{P}_1$ *(the value assigned by* $\mathcal{M}_1$*) otherwise.*

Let $[\![\_]\!]_i$ *be the interpretation generated by* $\mathcal{M}_i$ *for* $i = 0, 1, 2$*. Then, for any proposition* B*:*

– *if* B *contains only predicate and function symbols appearing in* A*,(remind that* $\top$ *and* $\bot$ *are connectives),* $[\![B]\!]_2 = [\![B]\!]_0$
– *if* B *contains no predicate or function symbol appearing in* A*,* $[\![B]\!]_2 = [\![B]\!]_1$

Proof. Easy induction on the structure of B. The base case is guaranteed by the definition and it propagates readily. □

Theorem 1. *Let* R *be a super-consistent rewrite system and let* A *be a closed* R*-compatible proposition.* R^A *is super-consistent.*

Proof. Let $\tilde{\mathcal{B}}$ be a pHA. Let $\mathcal{M}_0$ be any $\tilde{\mathcal{B}}$-valued structure, and $[\![\_]\!]_0$ the interpretation it generates. Let $a = [\![A]\!]_0$.

R has a $\tilde{\mathcal{B}}^a$-model because it is super-consistent. Let $[\![\_]\!]_1^a$ be the interpretation and $\mathcal{M}_1$ the associated $\tilde{\mathcal{B}}^a$-valued structure. $\mathcal{M}_1$ is as well a $\tilde{\mathcal{B}}$-valued structure, so let $\mathcal{M}_2$ be the A-grafting of $\mathcal{M}_0$ onto $\mathcal{M}_1$, as in Lemma 1. Let $[\![\_]\!]_2$ and $[\![\_]\!]_2^a$ be the interpretations generated in $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{B}}^a$, respectively. From Lemma 1 we derive:

– $[\![A]\!]_2 = [\![A]\!]_0$
– for any rewrite rule in R, $P \rightarrow F$, $[\![P]\!]_2^a = [\![P]\!]_1^a$ and $[\![F]\!]_2^a = [\![F]\!]_1^a$

In particular, $[\![\_]\!]_2^a$ inherits from $[\![\_]\!]_1^a$ the property to be a model of the rewrite system R. We have fulfilled the requirements of Proposition 6: the pHA is $\tilde{\mathcal{B}}$, the structure is $\mathcal{M}_2$, $[\![\_]\!]_2^a$ is a model of R in $\tilde{\mathcal{B}}^a = \tilde{\mathcal{B}}^{[\![A]\!]_2}$, since $[\![A]\!]_2 = [\![A]\!]_0 = a$.

Therefore R^A has a $\tilde{\mathcal{B}}$-model for any $\tilde{\mathcal{B}}$-model, and it is super-consistent. □

5 Super-consistency and Classical Sequent Calculus

5.1 From Intuitionistic to Classical Deduction Modulo

We adapt results of [10] to the settings of A-translation that shift cut-elimination in the intuitionistic calculus to the classical calculus. In the sequel we let R be a rewrite system and A be a closed R-compatible proposition.

Proposition 7. *Let* B, C *be propositions. If* $B \rightarrow_R C$ *then* $B^A \rightarrow_{R^A} C^A$*. If* $B \equiv_R C$ *then* $B^A \equiv_{R^A} C^A$*.*

Proof. By induction on the structure of B for the first point, and on the derivation of $B \equiv_R C$ for the second point. □

Proposition 8. *Assume that* A *is* R*-compatible. If* R *is a terminating and confluent rewrite system[19] then so is* R^A*.*

Proof. Consider a rewriting sequence $A_1 \rightarrow_{R^A} \cdots \rightarrow_{R^A} A_n$. A is R-compatible, so no proposition or term appearing in A can be rewritten. Thus we can define the rewriting sequence $A'_1 \rightarrow_R \cdots \rightarrow_R A'_n$, starting at $A'_1 = A_1$ by applying the same rules. This sequence must be finite.

As for confluence, consider a critical pair $C\ {}_{R^A}\!\leftarrow B \rightarrow_{R^A} D$, with B atomic. We know that B can be rewritten by the corresponding "antecedent" rules of R: $C_0\ {}_R\!\leftarrow B \rightarrow_R D_0$, with $C_0^A = C$ and $D_0^A = D$. Since R is confluent, there exists some proposition E_0 such that $C_0 \rightarrow^*_R E_0\ {}^*_R\!\leftarrow D_0$. We also have $C \rightarrow^*_R E_0^A\ {}^*_R\!\leftarrow D$ by Proposition 7, and R^A has the diamond property [19]. Since it is terminating, it is confluent. □

Lemma 2. *The rules* $\dfrac{\Gamma, C \vdash A}{\Gamma, C \Rightarrow A \Rightarrow A \vdash A}$ *and* $\dfrac{\Gamma \vdash C}{\Gamma, C \Rightarrow A \vdash A}$ *are derivable in intuitionistic sequent calculus modulo.*

Proof. Direct combination of $\Rightarrow$-l, $\Rightarrow$-r and axiom rules. □

Proposition 9. *If the sequent* $\Gamma \vdash \Delta$ *has a proof (with cuts) in the classical sequent calculus modulo* R *then* $\Gamma^A, (\Delta^A) \Rightarrow A \vdash A$ *has a proof (with cuts) in the intuitionistic sequent calculus modulo* R^A.

Proof. By an immediate induction we copy the structure of the proof of $\Gamma \vdash \Delta$, using Proposition 7 to rewrite propositions and the admissible rules of Lemma 2 to remove the tail As. This is the only hurdle to get back a sequent of a shape that allows us to apply the induction hypothesis.

Notice that, in the $\vee$-r case, we must apply once the $\vee_1$ rule and once the $\vee_2$, which requires a contraction on the left-hand side. □

Definition 15. *Let* $\Gamma \vdash \Delta; A$ *be an intuitionistic sequent.* Δ *contains at most one proposition and* $\Delta; A$ *stands for* A *if* Δ *is empty and* Δ *otherwise.*

$\Gamma \vdash \Delta; A$ *is said to represent a classical sequent* $A_1, \cdots, A_n \vdash B_1, \cdots, B_p$ *if there exists a one-to-one correspondence* ξ *between* $A_1, \cdots, A_n, B_1, \cdots B_p$ *and* Γ, Δ:

- *if* $\xi(A_i) \in \Gamma$ *then* $\xi(A_i) = A_i^A$ *or* $\xi(A_i) = A_i^A \Rightarrow A \Rightarrow A$
- *if* $\xi(A_i) \in \Delta$ *then* $\xi(A_i) = A_i^A \Rightarrow A$
- *if* $\xi(B_i) \in \Gamma$ *then* $\xi(B_i) = B_i^A \Rightarrow A$
- *if* $\xi(B_i) \in \Delta$ *then* $\xi(B_i) = B_i^A$ *or* $\xi(B_i) = B_i^A \Rightarrow A \Rightarrow A$

Lemma 3. *Let* B *be a proposition. Then* B^A *cannot be of the forms* A, $X \Rightarrow A$ *and* $X \Rightarrow A \Rightarrow A$.

Proof. A mere check of Definition 11 according to the structure of B. □

Proposition 10. *Let* A *be a proposition. Let* $\Gamma \vdash \Delta; A$ *be a sequent that represents* $A_1, \cdots, A_n \vdash B_1, \cdots, B_p$. *If this sequent has a cut-free proof in the intuitionistic sequent calculus modulo* R^A, and no right-rule other than axiom apply on A *then the sequent* $A_1, \cdots, A_n \vdash_R B_1, \cdots, B_p$ *has a cut-free proof in the classical sequent calculus modulo* R.

Proof. By induction on the intuitionistic proof of the sequent $\Gamma \vdash \Delta; A$, using Proposition 7:

- if the last rule is a logical rule applied to a proposition of the form A_i^A or B_i^A, we copy this rule and apply the induction hypothesis.
- If the last rule is a logical rule applied to a proposition of another form, it must be an $\Rightarrow$-l or a $\Rightarrow$-r rule. The sequent in the principal premiss is also a representation of the sequent $A_1, \cdots, A_n \vdash_R B_1, \cdots, B_p$ - potentially weakened by one proposition if Δ is not empty and a $\Rightarrow$-l rule was applied. So we just need to apply the induction hypothesis, potentially introducing a weak-r if necessary.
- if the last rule is an axiom, we copy it. Copying an axiom rule is possible because, by Lemma 3, the axiom rule can be only applied between propositions of the same nature, with no, a single, or two implications with A at the head and the same A-translated proposition at the base.
- if the last rule is a structural rule, we copy it on the side required by ξ and apply induction hypothesis. □

It is essential to assume that no rule apply on A other than axiom, otherwise the result fails; for instance the sequent $\vdash; C \Rightarrow C$ is intuitionistically provable while the empty sequent is not classically provable.

5.2 Cut Elimination in Classical Sequent Calculus Modulo

Theorem 2. *If a rewrite system R is super-consistent the classical sequent calculus modulo R has the cut elimination property.*

Proof. Let $\Gamma \vdash \Delta$ be a provable sequent in the classical sequent calculus modulo R. Let A be a proposition not containing any predicate or function symbol of R. The sequent $\Gamma^A, \Delta^A \Rightarrow A \vdash A$ has a proof in the intuitionistic sequent calculus modulo R^A by Proposition 9 above. By Theorem 1, R^A is super-consistent. Therefore, by Corollary 4.1 of Proposition 4.1 of, $\Gamma^A, \Delta^A \Rightarrow A \vdash A$ has a cut-free proof in the intuitionistic sequent calculus.

Moreover, no rule on A other than axiom is introduced: Proposition 9 introduces only axioms, that are translated into axioms in natural deduction, and the structure of A is therefore not exposed to any introduction or elimination rules. Another argument is that we can "freeze" A and view it as an atomic formula in all the discussion above. So the proof cannot use any information on A, since it is a generic parameter of the theorem.

Consequently, by Proposition 10 the sequent $\Gamma \vdash \Delta$ has a cut-free proof. □

Note that the argument appeals to a normalization procedure of the proof-terms of Natural deduction modulo, considering commutative cuts (Section 3.6 of [10]). Other cut elimination methods for Natural deduction modulo (as the one of [9]) do not apply since they do not get rid of commutative cuts.

6 Conclusion

In [10] $R^\perp$ had to be assumed to have a pre-model in order to show cut elimination for the classical sequent calculus modulo R (Theorem 4.1 of [10]). [7] shows that it is sufficient to show $R^\perp$ to be super-consistent. We have shown here that we can instead discuss the super-consistency of R directly.

Our result is a priori more restrictive, since by instantiating A by $\perp$ we get the super-consistency of $R^\perp$ that in turn implies the existence of a pre-model for $R^\perp$. It is currently unknown whether all those criteria are equivalent or not: can we, for instance, find a rewrite system and a proposition A, such that R^A is super-consistent while R is not super consistent ? Does the existence of a pre-model for R entail super-consistency ? On the good side, our criterion works directly on R and avoids a duplication of arguments: we now in one pass have normalization for natural deduction modulo R ([7,10]) and cut elimination for the classical sequent calculus, and bypass the need of two separate pre-model (or super-consistency arguments) for R and $R^\perp$. Moreover, super-consistency, by abstracting over reducibility candidates, provides a certain ease of use.

We have also shown a general result, by A-translating rewrite systems and semantics frameworks, instead of $\perp$-translating them. For the proof of cut elimination, we believe that the latter, better known as double-negation translation, would have been sufficient, as in [10]. But the work on A-translation bears a more general character, that can be used for other applications.

Super-consistency appears to be the right criterion to deal with when one wants to know about the cut elimination property of a deduction modulo theory, as the property holds whatever the syntactic calculus is. It would be interesting to see how the super-consistency criterion extends to other first-order framework, like the calculus of structures [15] or $\lambda\Pi$-calculus modulo, that is at the root of the Dedukti proof-checker [1].

Whether we can widen the criterion and replace pseudo-Heyting algebras by Heyting algebras in Definition 10, the idea being to use *cut-admissibility* (through semantic completeness, in the mood of [17] for instance) instead of normalization in the proof of Theorem 2 is a conjecture. Analyzing [4,9] closely shows that cut-admissibility results crucially depend on finding in the interpretation of the atoms P a syntactical version of P in the model formed out of contexts/propositions. Super-consistency does not *directly* allows this, due to the abstract construction of a generic model. This appeals to a more informative structure, in both papers algebras of sequents were introduced which happens to be only pseudo-Heyting algebras.

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