Double Negation Translations as Morphisms

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Double-Negation Translations

Double-Negation translations:

- a shallow way to encode classical logic into intuitionistic
- Zenon's backend for Dedukti
- existing translations: Kolmogorov's (1925), Gentzen-Gödel's (1933), Kuroda's (1951), Krivine's (1990), $\cdots$

Minimizing the translations:

- turns more formulæ into themselves;
- shifts a classical proof into an intuitionistic proof of the *same* formula.

Morphisms

- A morphism preserves the operations between two structures:

$$\text{Group morphism: } \begin{cases} (\mathbb{Z}, +, 0) & \mapsto & (\mathbb{R}^*, *, 1) \\ h(0) & \to & 1 \\ h(a+b) & \to & h(a) * h(b) \end{cases}$$

- a translation that is a morphism:

$$\begin{array}{rcl} h(P) & = & P \\ h(A \wedge B) & = & h(A) \wedge h(B) \\ h(A \vee B) & = & h(A) \vee h(B) \\ h(A \Rightarrow B) & = & h(A) \Rightarrow h(B) \\ h(\forall x A) & = & \forall\ x\ h(A) \\ h(\exists x A) & = & \exists\ x\ h(A) \end{array}$$

(of course this is the identity)

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- a **more interesting** translation that is a morphism:

$$\begin{array}{rcl} h(P) & = & P \\ h(A \wedge B) & = & h(A) \wedge_c h(B) \\ h(A \vee B) & = & h(A) \vee_c h(B) \\ h(A \Rightarrow B) & = & h(A) \Rightarrow_c h(B) \\ h(\forall x A) & = & \forall_c x\, h(A) \\ h(\exists x A) & = & \exists_c x\, h(A) \end{array}$$

two kinds of connectives: the **classical** and the **intuitionistic** ones.

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two kinds of connectives: the **classical** and the **intuitionistic** ones.

- Design a unified logic, where we can reason both classically and intuitionistically:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee_i B} \qquad \frac{\text{strange premises}}{\Gamma \vdash A \vee_c B}$$

Translations that are Morphisms

- None of the previous translations is a morphism.
- Dowek has shown one, it is very verbose.
- We make it lighter.

Plan:

1. Classical and Intuitionistic Logic
2. Sequent Calculus
3. Double Negation Translations
4. Morphisms

Classical vs. Intuitionistic

- The principle of excluded-middle. Should

$$A \vee \neg A$$

be provable ? Yes or no ?

- Yes. This is what is called classical logic.
- Wait a minute !

The Drinker's Principle

In a bar, there is somebody such that, if he drinks, then everybody drinks.

Two Irrationals

There exists $i_1, i_2 \in \mathbb{R} \backslash \mathbb{Q}$ such that $i_1^{i_2} \in \mathbb{Q}$.

IRL: A Manicchean World

You are with us, or against us (G. W. Bush)
Rashomon (A. Kurosawa)

Classical vs. Intuitionistic

- The principle of excluded-middle. Should

$$A \vee \neg A$$

be provable ? Yes or no ?

- No. This is the constructivist school (Brouwer, Heyting, Kolmogorov).
- Intuitionistic logic is one of those branches. It features the BHK interpretation of proofs:

Witness Property

A proof of $\exists x A$ (in the empty context) gives a witness t for the property A.

Disjunction Property

A proof of $A \vee B$ (in the empty context) reduces eventually either to a proof of A, or to a proof of B.

The Classical Sequent Calculus (LK)

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \text{ax}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_L \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee_R$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ \Rightarrow_L \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ \Rightarrow_R$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \neg_L \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg_R$$

$$\frac{\Gamma, A[c/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}\ \exists_L \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta}\ \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}\ \forall_L \qquad \frac{\Gamma \vdash A[c/x], \Delta}{\Gamma \vdash \forall x A, \Delta}\ \forall_R$$

The Intuitionistic Sequent Calculus (LJ)

$$\frac{}{\Gamma, A \vdash A}\ \text{ax}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_L \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee_{R1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee_{R2}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ \Rightarrow_L \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ \Rightarrow_R$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash \Delta}\ \neg_L \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ \neg_R$$

$$\frac{\Gamma, A[c/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}\ \exists_L \qquad \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A}\ \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}\ \forall_L \qquad \frac{\Gamma \vdash A[c/x]}{\Gamma \vdash \forall x A}\ \forall_R$$

Note on Frameworks

- structural rules are not shown (contraction, weakening)
- left-rules seem very similar in both cases
- so, lhs formulæ can be translated by themselves
- this accounts for polarizing the translations
- another work [Boudard & H]:
  - ⋆ does not behave well in presence of cuts
  - ⋆ appeals to focusing techniques

Examples

- proofs that behave identically in classical/intuitionistic logic:

$$\dfrac{\dfrac{}{A, B \vdash A}\,\text{ax}}{A \vdash B \Rightarrow A}\Rightarrow_R \qquad \dfrac{\dfrac{\dfrac{}{A, B \vdash B}\,\text{ax}}{A \wedge B \vdash B}\wedge_L}{A \wedge B \vdash B \vee C}\vee_R$$

- proof of the excluded-middle:

Classical Logic	Intuitionistic Logic
$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{ax}}{\vdash A, \neg A}\neg_R}{\vdash A \vee \neg A}\vee_R$	

Examples

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$$\dfrac{\dfrac{}{A, B \vdash A}\,ax}{A \vdash B \Rightarrow A}\Rightarrow_R \qquad \dfrac{\dfrac{\dfrac{}{A, B \vdash B}\,ax}{A \wedge B \vdash B}\wedge_L}{A \wedge B \vdash B \vee C}\vee_R$$

- proof of the excluded-middle:

Classical Logic	Intuitionistic Logic
$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax}{\vdash A, \neg A}\neg_R}{\vdash A \vee \neg A}\vee_R$	$\dfrac{??}{\vdash A \vee \neg A}$

The Excluded-Middle in Intuitionistic Logic

- is not provable. However, its negation is inconsistent.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\text{ax}}{A \vdash A \vee \neg A}\vee_{R1}}{\neg(A \vee \neg A), A \vdash}\neg_L}{\neg(A \vee \neg A) \vdash \neg A}\neg_R}{\neg(A \vee \neg A) \vdash A \vee \neg A}\vee_{R2}}{\neg(A \vee \neg A), \neg(A \vee \neg A) \vdash}\neg_L}{\neg(A \vee \neg A) \vdash}\text{contraction}$$

The Excluded-Middle in Intuitionistic Logic

- is not provable. However, its negation is inconsistent.
- this suggests a scheme for a translation between int. and clas. logic:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{ax}}{\vdash A, \neg A}\,\neg_R}{\vdash A \vee \neg A}\,\vee_R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{ax}}{A \vdash A \vee \neg A}\,\vee_{R1}}{\neg(A \vee \neg A), A \vdash}\,\neg\mathbf{L}}{\neg(A \vee \neg A) \vdash \neg A}\,\neg_R}{\neg(A \vee \neg A) \vdash A \vee \neg A}\,\vee_{R2}}{\neg(A \vee \neg A), \neg(A \vee \neg A) \vdash}\,\neg\mathbf{L}}{\neg(A \vee \neg A) \vdash}\,\text{contr}}{}$$

- need: $\neg\neg$ everywhere in Δ (and Γ)
- the proof of the “negation of the excluded middle” requires duplication (contraction), which partly explain why we allow several formulæ on the rhs in LK.

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- this suggests a scheme for a translation between int. and clas. logic:

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- given a classical proof $\Gamma \vdash \Delta$, store Δ on the lhs, and translate:

Clas. | Int.

$$\text{rule } r\ \dfrac{\Gamma \vdash A_1, \Delta \qquad \Gamma \vdash A_2, \Delta}{\Gamma \vdash A, \Delta} \qquad \text{rule } r\ \dfrac{\dfrac{\dfrac{\Gamma, \neg A_1, \neg\Delta \vdash}{\Gamma, \neg\Delta \vdash \neg\neg A_1}\,\neg\mathbf{R} \qquad \dfrac{\Gamma, \neg A_2, \neg\Delta \vdash}{\Gamma, \neg\Delta \vdash \neg\neg A_2}\,\neg\mathbf{R}}{\Gamma, \neg\Delta \vdash A}}{\Gamma, \neg\Delta, \neg A \vdash}\,\neg\mathbf{L}$$

- need: $\neg\neg$ everywhere in Δ (and Γ)
- the proof of the "negation of the excluded middle" requires duplication (contraction), which partly explain why we allow several formulæ on the rhs in LK.

Kolmogorov's Translation

Kolmogorov's $\neg\neg$-translation introduces $\neg\neg$ everywhere:

$$
\begin{array}{rcll}
B^{Ko} & = & \neg\neg B & \text{(atoms)} \\
(B \wedge C)^{Ko} & = & \neg\neg(B^{Ko} \wedge C^{Ko}) & \\
(B \vee C)^{Ko} & = & \neg\neg(B^{Ko} \vee C^{Ko}) & \\
(B \Rightarrow C)^{Ko} & = & \neg\neg(B^{Ko} \Rightarrow C^{Ko}) & \\
(\forall x A)^{Ko} & = & \neg\neg(\forall x A^{Ko}) & \\
(\exists x A)^{Ko} & = & \neg\neg(\exists x A^{Ko}) &
\end{array}
$$

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^{Ko}, \lrcorner\Delta^{Ko} \vdash$ is provable in LJ.

Antinegation

$\lrcorner$ is an operator, such that:

- $\lrcorner\neg A = A$;
- $\lrcorner B = \neg B$ otherwise.

Light Kolmogorov's Translation

Moving negation from connectives to formulæ [Dowek& Werner]:

$$\begin{array}{rcll} B^K & = & B & \text{(atoms)} \\ (B \wedge C)^K & = & (\neg\neg B^K \wedge \neg\neg C^K) & \\ (B \vee C)^K & = & (\neg\neg B^K \vee \neg\neg C^K) & \\ (B \Rightarrow C)^K & = & (\neg\neg B^K \Rightarrow \neg\neg C^K) & \\ (\forall x A)^K & = & \forall x \neg\neg A^K & \\ (\exists x A)^K & = & \exists x \neg\neg A^K & \end{array}$$

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^K, \neg\Delta^K \vdash$ is provable in LJ.

Correspondence

$A^{Ko} = \neg\neg A^K$

How does the Translation Work ?

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^K, \neg\Delta^K \vdash$ is provable in LJ.

Proof: Induction on the LK proof. $\neg$ bounces. Example: rule $\wedge_R$.

$$\wedge_R \dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}$$

is turned into:

$$\dfrac{\dfrac{\pi_1'}{\Gamma^K, \neg A^K, \neg\Delta^K \vdash} \quad \dfrac{\pi_2'}{\Gamma^K, \neg B^K, \neg\Delta^K \vdash}}{\Gamma^K, \neg(\neg\neg A^K \wedge \neg\neg B^K), \neg\Delta^K \vdash} \wedge_R$$

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$$\wedge_R\ \dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}$$

is turned into:

$$\neg_L\ \dfrac{\dfrac{\dfrac{\pi'_1}{\Gamma^K, \neg A^K, \neg\Delta^K \vdash} \quad \dfrac{\pi'_2}{\Gamma^K, \neg B^K, \neg\Delta^K \vdash}}{\Gamma^K, \neg\Delta^K \vdash \neg\neg A^K \wedge \neg\neg B^K}\ \wedge_R}{\Gamma^K, \neg(\neg\neg A^K \wedge \neg\neg B^K), \neg\Delta^K \vdash}$$

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is turned into:

$$\neg_L \dfrac{\wedge_R \dfrac{\neg_R \dfrac{\dfrac{\pi'_1}{\Gamma^K, \neg A^K, \neg\Delta^K \vdash}}{\Gamma^K, \neg\Delta^K \vdash \neg\neg A^K} \quad \neg_R \dfrac{\dfrac{\pi'_2}{\Gamma^K, \neg B^K, \neg\Delta^K \vdash}}{\Gamma^K, \neg\Delta^K \vdash \neg\neg B^K}}{\Gamma^K, \neg\Delta^K \vdash \neg\neg A^K \wedge \neg\neg B^K}}{\Gamma^K, \neg(\neg\neg A^K \wedge \neg\neg B^K), \neg\Delta^K \vdash}$$

Are they morphisms ?

Consider Kolmogorov's translation:

- let:

$$\begin{aligned} B \wedge_c C &= \neg\neg(B \wedge_i C) \\ B \vee_c C &= \neg\neg(B \vee_i C) \\ B \Rightarrow_c C &= \neg\neg(B \Rightarrow_i C) \\ \forall_c x A &= \neg\neg(\forall_i x A) \\ \exists_c x A &= \neg\neg(\exists_i x A) \end{aligned}$$

- unfortunately:

$$\begin{aligned} B^{Ko} &= \neg\neg B \qquad \text{(atoms)} \\ (B \wedge C)^{Ko} &= B^{Ko} \wedge_c C^{Ko} \\ (B \vee C)^{Ko} &= B^{Ko} \vee_c C^{Ko} \\ (B \Rightarrow C)^{Ko} &= B^{Ko} \Rightarrow_c C^{Ko} \\ (\forall x A)^{Ko} &= \forall_c x A^{Ko} \\ (\exists x A)^{Ko} &= \exists_c x A^{Ko} \end{aligned}$$

- this is not a morphism.

Are they morphisms ?

- No !
  - in the case of *Ko*:

$$B^{Ko} = \neg\neg B(\text{atoms})$$

  - in the case of *K* :

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^K, \neg\Delta^K \vdash$ is provable in LJ.

  - exercise: these negations are necessary (hint: consider the excluded-middle and its derivatives)
- can we be more clever ?
  - some intuitionistic right-rules are the same as classical right-rules. For instance, $\wedge_R$:

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}$$

  - Translate them by themselves. Gödel-Gentzen translation.

Gödel-Gentzen Translation

In this translation, disjunctions and existential quantifiers are replaced by a combination of negation and their De Morgan duals:

$$\begin{aligned}
B^{gg} &= \neg\neg B \\
(A \wedge B)^{gg} &= A^{gg} \wedge B^{gg} \\
(A \vee B)^{gg} &= \neg(\neg A^{gg} \wedge \neg B^{gg}) \\
(A \Rightarrow B)^{gg} &= A^{gg} \Rightarrow B^{gg} \\
(\forall x A)^{gg} &= \forall x A^{gg} \\
(\exists x A)^{gg} &= \neg\forall x \neg A^{gg}
\end{aligned}$$

Example of translation

$((A \vee B) \Rightarrow C)^{gg}$ is $(\neg(\neg\neg\neg A \wedge \neg\neg\neg B)) \Rightarrow \neg\neg C$

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^{gg}, \lrcorner\Delta^{gg} \vdash$ is provable in LJ.

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- No !
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  - in the case of *K* :

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^K, \neg\Delta^K \vdash$ is provable in LJ.

  - exercise: show that those negations are necessary (hint: consider the excluded-middle and its derivatives)
- can we be more clever ?
  - some intuitionistic right-rules are the same as classical right-rules. For instance, $\wedge_R$:

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}$$

  - Gödel-Getzen translation:
  - is still not a morphism !
- etc. for all the other known translations (Krivine, Kuroda, Frédéric Gilbert)

How to make a morphism: an analysis

- Translation of, say, $A \wedge B$:

Kolmogorov	Light Kolmogorov
$\neg\neg(A^{Ko} \wedge B^{Ko})$	$(\neg\neg A^{Ko}) \wedge (\neg\neg A^{Ko})$

- Feature, double-negation:

Kolmogorov	Light Kolmogorov
on top of connectives	inside connectives

- Analysis, problem appearing in:

	Kolmogorov	Light Kolmogorov
Problem	atoms: $\neg\neg P$	statement: $\Gamma^{K}, \neg\Delta^{K} \vdash$
Solution	statement: $\Gamma^{Ko}, \lrcorner\Delta^{Ko} \vdash$	atoms: P

- Solution: combine them !

Dowek's translation

$$
\begin{array}{rcl}
B^D & = & B & = & B & \text{(atoms)} \\
(B \wedge C)^D & = & B^D \wedge_c C^D & = & \neg\neg(\neg\neg B^D \wedge \neg\neg C^D) \\
(B \vee C)^D & = & B^D \vee_c C^D & = & \neg\neg(\neg\neg B^D \vee \neg\neg C^D) \\
(B \Rightarrow C)^D & = & B^D \Rightarrow_c C^D & = & \neg\neg(\neg\neg B^D \Rightarrow \neg\neg C^D) \\
(\forall x A)^D & = & \forall_c x A^D & = & \neg\neg\forall x \neg\neg A^D \\
(\exists x A)^D & = & \exists_c x A^D & = & \neg\neg\exists x \neg\neg A^D
\end{array}
$$

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^D, \lrcorner\Delta^D \vdash$ is provable in LJ.

Corollary

Assume A is not atomic. $\Gamma \vdash A$ is provable in LK iff $\Gamma^D \vdash A^D$ is provable in LJ.

Proof:

- $\neg\lrcorner A^D = A^D$ (except in the atomic case) □

The Price to Pay

- heavy: for each connective, 6 negations. $((A \vee B) \Rightarrow C)^D$ is $\neg\neg(\neg\neg\neg\neg(\neg\neg A \vee \neg\neg B) \Rightarrow \neg\neg C)$
- most of the time useless, except at the top and at the bottom of the formula
- remember Gödel-Gentzen's idea. Use De Morgan duals:

$$(A \vee B)^{gg} = \neg(\neg A^{gg} \vee \neg B^{gg})$$

- let us do the same, and divide by two the number of double negations.

A Light Morphism

Remember De Morgan,

$$
\begin{aligned}
A \vee B &= \neg(\neg A \wedge \neg B) \\
A \wedge B &= \neg(\neg A \vee \neg B) \\
A \Rightarrow B &= \neg A \vee B \\
\neg A &= \neg A \\
\forall x A &= \neg \exists x \neg A \\
\exists x A &= \neg \forall x \neg A
\end{aligned}
$$

A Light Morphism

Remember De Morgan, and let

$$
\begin{array}{rcl}
A \vee_c B & = & \neg(\neg A \wedge \neg B) \\
A \wedge_c B & = & \neg(\neg A \vee \neg B) \\
A \Rightarrow_c B & = & \neg(\neg\neg A \vee \neg B \\
\neg_c A & = & \neg\neg\neg A \\
\forall_c x A & = & \neg\exists x \neg A \\
\exists_c x A & = & \neg\forall x \neg A
\end{array}
$$

- this gives rise to a morphism, $(.)^\circ$ together with:

$$
\begin{array}{rcl}
\top_c & = & \neg\neg\top \\
\bot_c & = & \neg\neg\bot
\end{array}
$$

- and we can prove the theorem:

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^\circ, \lrcorner\Delta^\circ \vdash$ is provable in LJ.

Some Cases

Proof by induction on the proof of $\Gamma \vdash \Delta$.

- last rule $\vee_R$ on some $A \vee B \in \Delta$. Remember: $\lrcorner(A \vee B)^\circ = \neg A^\circ \wedge \neg B^\circ$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}$$

  - ★ A and B are atomic: $\lrcorner A^\circ = \neg A$ and $\lrcorner B^\circ = \neg B$.

$$\frac{\Gamma^\circ, \neg A, \neg B, \lrcorner\Delta^\circ \vdash}{\Gamma^\circ, \neg A \wedge \neg B, \lrcorner\Delta^\circ \vdash}$$

  - ★ if neither A and B are atomic, then A° and B° have a trailing $\neg$, and we remove it (bouncing):

$$\frac{\dfrac{\Gamma^\circ, \lrcorner A^\circ, \lrcorner B^\circ, \lrcorner\Delta^\circ \vdash}{\Gamma^\circ, \neg A^\circ, \neg B^\circ, \lrcorner\Delta^\circ \vdash}\,(\neg_R, \neg_L) \text{ x } 2}{\Gamma^\circ, \neg A^\circ \wedge \neg B^\circ, \lrcorner\Delta^\circ \vdash}$$

  - ★ mixed case: mixed strategy.

Conclusion

- logic with two kinds of connectives: $\vee_i$ and $\vee_c$

$$\vee_{R1}\ \frac{\Gamma \vdash A}{\Gamma \vdash A \vee_i B} \qquad \vee_{R2}\ \frac{\Gamma \vdash B}{\Gamma \vdash A \vee_i B} \qquad \vee_{cR}\ \frac{\Gamma, \neg A, \neg B \vdash}{\Gamma \vdash A \vee_c B}$$

- and we have:

I

f Γ, Δ, A contain only classical connectives, A non atomic, then $\Gamma \vdash A$ in LK iff $\Gamma \vdash A$. As well, $\Gamma \vdash \Delta$ in LK iff $\Gamma, \lrcorner\Delta \vdash$.

- Getting lighter morphisms:
  - ★ from $\neg_c A = \neg\neg\neg A$ to $\neg_c A = \neg A$?
  - ★ from $A \Rightarrow_c B = \neg(\neg\neg A \vee \neg B$ to $A \Rightarrow_c B = \neg(A \vee \neg B)$?
  - ★ we cannot always maintain the invariant $\Gamma, \lrcorner\Delta \vdash$.
  - ★ Focusing in LK to the rescue.

Lighter Morphisms

Remember De Morgan, and let:

$$
\begin{array}{rcl}
A \vee_c B & = & \neg(\neg A \wedge \neg B) \\
A \wedge_c B & = & \neg(\neg A \vee \neg B) \\
A \Rightarrow_c B & = & \neg(\neg\neg A \vee \neg B \\
\neg_c A & = & \neg\neg\neg A \\
\forall_c x A & = & \neg\exists x \neg A \\
\exists_c x A & = & \neg\forall x \neg A
\end{array}
$$

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\neg_c A & = & \neg \; A \\
\forall_c x A & = & \neg\exists x \neg A \\
\exists_c x A & = & \neg\forall x \neg A
\end{array}$$

- this gives rise to a morphism, $(.)^\circ$ together with:

$$\begin{array}{rcl}
\top_c & = & \top \\
\bot_c & = & \bot
\end{array}$$

- and we can prove the theorem:

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^\circ, \lrcorner\Delta^\circ \vdash$ is provable in LJ.

Dismissing Cuts ?

The cut rule:

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\text{cut}$$

- connects rhs and lhs
- dual to the axiom rule:

$$\frac{}{\Gamma, A \vdash A, \Delta}\text{axiom}$$

- is admissible/eliminable (Gentzen)

If we do not have connections, then we can differentiate rhs and lhs translations:

$$\begin{array}{rclcrcl}
A \vee_c^l B & = & A \vee B & & A \vee_c^r B & = & \neg(\neg A \wedge \neg B) \\
A \wedge_c^l B & = & A \wedge B & & A \wedge_c^r B & = & \neg(\neg A \vee \neg B) \\
A \Rightarrow_c^l B & = & A \Rightarrow B & & A \Rightarrow_c^r B & = & \neg(\neg\neg A \vee \neg B \\
\neg_c^l A & = & \neg A & & \neg_c^r A & = & \neg\neg\neg A \\
\forall_c^l xA & = & \forall xA & & \forall_c^r xA & = & \neg\exists x\neg A \\
\exists_c^l xA & = & \exists xA & & \exists_c^r xA & = & \neg\forall x\neg A
\end{array}$$

A Focus on LK → LJ

- less negations imposes more discipline. Example:

$$\wedge_R \dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \qquad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta} \quad \text{becomes} \quad \neg_L \dfrac{\wedge_R \dfrac{??\dfrac{\dfrac{\pi_1'}{\Gamma^l, \lrcorner A^r, \lrcorner\Delta^r \vdash}}{\Gamma^l, \lrcorner\Delta^r \vdash A^r} \qquad \dfrac{\dfrac{\pi_2'}{\Gamma^l, \lrcorner B^r, \lrcorner\Delta^r \vdash}}{\Gamma^l, \lrcorner\Delta^r \vdash B^r}??}{\Gamma^l, \lrcorner\Delta^r \vdash A^r \wedge B^r}}{\Gamma^l, \neg(A^r \wedge B^r), \lrcorner\Delta^r \vdash}$$

- when A^r introduces negations ($\exists, \vee, \neg$ and atomic cases) ?? can be $\neg_R$ due to the behavior of $\lrcorner A^r$
- otherwise A^r remains of the rhs in the LJ proof.

A Focus on LK → LJ

- less negations imposes more discipline. Example:

$$\dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \qquad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}\wedge_R \quad \text{becomes} \quad \dfrac{\dfrac{\dfrac{\dfrac{\pi'_1}{\Gamma^l, \lrcorner A^r, \lrcorner\Delta^r \vdash}}{\Gamma^l, \lrcorner\Delta^r \vdash A^r}?? \qquad \dfrac{\dfrac{\pi'_2}{\Gamma^l, \lrcorner B^r, \lrcorner\Delta^r \vdash}}{\Gamma^l, \lrcorner\Delta^r \vdash B^r}??}{\Gamma^l, \lrcorner\Delta^r \vdash A^r \wedge B^r}\wedge_R}{\Gamma^l, \neg(A^r \wedge B^r), \lrcorner\Delta^r \vdash}\neg_L$$

- when A^r introduces negations ($\exists, \vee, \neg$ and atomic cases) ?? can be $\neg_R$ due to the behavior of $\lrcorner A^r$
- otherwise A^r remains of the rhs in the LJ proof.
- the next rule in π_1 and π_2 must be on A (resp. B). How ?

A Focus on LK $\rightarrow$ LJ

- less negations imposes more discipline. Example:

$$
\wedge_R \; \dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \qquad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}
\quad \text{becomes} \quad
\neg_L \; \dfrac{\dfrac{??\;\dfrac{\dfrac{\pi'_1}{\Gamma^l, \lrcorner A^r, \lrcorner \Delta^r \vdash}}{\Gamma^l, \lrcorner \Delta^r \vdash A^r} \qquad \dfrac{\dfrac{\pi'_2}{\Gamma^l, \lrcorner B^r, \lrcorner \Delta^r \vdash}}{\Gamma^l, \lrcorner \Delta^r \vdash B^r}\;??}{\Gamma^l, \lrcorner \Delta^r \vdash A^r \wedge B^r}\;\wedge_R}{\Gamma^l, \neg(A^r \wedge B^r), \lrcorner \Delta^r \vdash}
$$

- when A^r introduces negations ($\exists, \vee, \neg$ and atomic cases) ?? can be $\neg_R$ due to the behavior of $\lrcorner A^r$
- otherwise A^r remains of the rhs in the LJ proof.
- the next rule in π_1 and π_2 must be on A (resp. B). How ?
- use Kleene's inversion lemma
- or ... this is exactly what focusing is about !

A Focused Classical Sequent Calculus

Sequent with focus

A focused sequent $\Gamma \vdash A; \Delta$ has three parts:

- Γ and Δ
- A, the (possibly empty) stoup formula

$$\Gamma \vdash \underbrace{\quad . \quad}_{\text{stoup}}; \Delta$$

- when the stoup is not empty, the next rule must apply on its formula,
- under some conditions, it is possible to move/remove a formula in/from the stoup.

A Focused Sequent Calculus

$$\frac{}{\Gamma, A \vdash \ .\ ;\ A, \Delta}\ \text{ax}$$

$$\frac{\Gamma, A, B \vdash \ .\ ;\ \Delta}{\Gamma, A \wedge B \vdash \ .\ ;\ \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A\ ;\ \Delta \qquad \Gamma \vdash B\ ;\ \Delta}{\Gamma \vdash A \wedge B\ ;\ \Delta}\ \wedge_R$$

$$\frac{\Gamma, A \vdash \ .\ ;\ \Delta \qquad \Gamma, B \vdash \ .\ ;\ \Delta}{\Gamma, A \vee B \vdash \ .\ ;\ \Delta}\ \vee_L \qquad \frac{\Gamma \vdash \ .\ ;\ A, B, \Delta}{\Gamma \vdash \ .\ ;\ A \vee B, \Delta}\ \vee_R$$

$$\frac{\Gamma \vdash A\ ;\ \Delta \qquad \Gamma, B \vdash \ .\ ;\ \Delta}{\Gamma, A \Rightarrow B \vdash \ .\ ;\ \Delta}\ \Rightarrow_L \qquad \frac{\Gamma, A \vdash B\ ;\ \Delta}{\Gamma \vdash A \Rightarrow B\ ;\ \Delta}\ \Rightarrow_R$$

$$\frac{\Gamma, A[c/x] \vdash \ .\ ;\ \Delta}{\Gamma, \exists x A \vdash \ .\ ;\ \Delta}\ \exists_L \qquad \frac{\Gamma \vdash \ .\ ;\ A[t/x], \Delta}{\Gamma \vdash \ .\ ;\ \exists x A, \Delta}\ \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash \ .\ ;\ \Delta}{\Gamma, \forall x A \vdash \ .\ ;\ \Delta}\ \forall_L \qquad \frac{\Gamma \vdash A[c/x]\ ;\ \Delta}{\Gamma \vdash \forall x A\ ;\ \Delta}\ \forall_R$$

$$\frac{\Gamma \vdash A\ ;\ \Delta}{\Gamma \vdash \ .\ ;\ A, \Delta}\ \text{focus} \qquad \frac{\Gamma \vdash \ .\ ;\ A, \Delta}{\Gamma \vdash A\ ;\ \Delta}\ \text{release}$$

A Focused Sequent Calculus

$$\frac{\Gamma \vdash A \,;\, \Delta}{\Gamma \vdash \,.\,;\, A, \Delta} \text{ focus} \qquad \frac{\Gamma \vdash \,.\,;\, A, \Delta}{\Gamma \vdash A \,;\, \Delta} \text{ release}$$

Characteristics:

- in release, A is either atomic or of the form $\exists x B, B \vee C$ or $\neg B$;
- in focus, the converse holds: A must not be atomic, nor of the form $\exists x B, B \vee C$ nor $\neg B$.
- the *synchronous* (outside the stoup) right-rules are $\exists_R$, $\neg_R$, $\vee_R$ and (atomic) axiom: the exact places where $\{.\}^r$ introduces negation

Theorem

If $\Gamma \vdash \Delta$ is provable in LK then $\Gamma \vdash .; \Delta$ is provable.

Proof: use Kleene's inversion lemma (holds for all connectives/quantifiers, except $\exists_R$ and $\forall_L$).

Translating Focused Proofs in LJ

$$\frac{\Gamma \vdash \; A \; ; \Delta}{\Gamma \vdash \; . \; ; A, \Delta} \text{focus} \qquad \frac{\Gamma \vdash \; . \; ; A, \Delta}{\Gamma \vdash \; A \; ; \Delta} \text{release}$$

Theorem

If $\Gamma \vdash A; \Delta$ in focused LK, then $\Gamma^l, \lrcorner\Delta^r \vdash A^r$ in LJ

- release is translated by the $\neg_R$ rule
- focus is translated by the $\neg_L$ rule

Translating Focused Proofs in LJ

$$\frac{\Gamma \vdash A ; \Delta}{\Gamma \vdash . ; A, \Delta} \text{ focus} \qquad \frac{\Gamma \vdash . ; A, \Delta}{\Gamma \vdash A ; \Delta} \text{ release}$$

Theorem

If $\Gamma \vdash A; \Delta$ in focused LK, then $\Gamma^l, \lrcorner\Delta^r \vdash A^r$ in LJ

- release is translated by the $\neg_R$ rule
- focus is translated by the $\neg_L$ rule
- $\lrcorner\Delta^r$ removes the trailing negation on $\exists^n$ $(\neg\forall\neg)$, $\vee^r$ $(\neg \wedge \neg)$, $\neg^r$ $(\neg)$ and atoms $(\neg\neg)$
- what a surprise: focus is forbidden on them, so rule on the lhs:

LK rule	$\exists_R$	$\neg_R$	$\vee_R$	ax.
LJ rule	$\forall_L$	nop	$\wedge_L$	$\neg_L$ + ax.

- see the paper.