Complétude en logiques

Habilitation à diriger des recherches

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Completeness in Logics

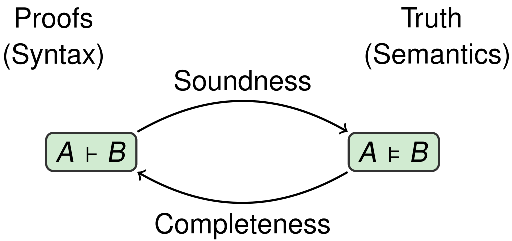
Major role of the Completeness Theorem:

- Gödel sense
- exhaustive proof-search succeeds (eventually)
- fundamental property: cut elimination

Logics:

- automated theorem proving [P. Halmagrand]
- proof checking [R. Saillard]
- application domain: formal methods
  - large (mathematical) proofs
  - safe, *bug-free*, system conception
- theory of programming languages (type systems, semantics, static analysis) [T. Giang-Le, V. Maisonneuve]
- model checking, realizability, ...

Key Properties of Logical Systems



Theorem (Soundness)

If a statement is provable, it is (universally) true.

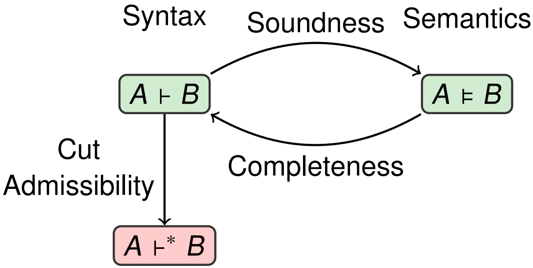
Corollary (Consistency)

Not all statements have proofs.

Theorem (Completeness)

If a statement is (universally) true, it is provable.

Key Properties of Logical Systems



Theorem (Cut Admissibility)

If a statement is provable, then it is provable without cut.

- consistency
- automated proof-search
- focus on computation (CS point of view):
  - the site for interaction
  - proof terms
  - normalization (termination of proof-term reduction)

Outline

Propositional Logic

- atomic formulas, connectives $\wedge, \vee, \Rightarrow, \neg, \bot, \top$
- semantics, truth tables

A	B	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$\neg A$	$\bot$	$\top$
0	0	0	0	1	1	0	1
0	1	0	1	1	1	0	1
1	0	0	1	0	0	0	1
1	1	1	1	1	0	0	1

- syntax, a proof-search method called *the tableaux method*.
- refutation-based method: to show F, derive a contradiction from $\neg F$.
- one rule per connective, another for its negation

$$\frac{\bot}{\odot}\,\bot \qquad \frac{\neg\top}{\odot}\,\neg_\top \qquad \frac{F, \neg F}{\odot}\,\text{cl}$$

$$\frac{A \wedge B}{A, B}\,\wedge \qquad \frac{\neg(A \vee B)}{\neg A, \neg B}\,\neg_\vee \qquad \frac{\neg(A \Rightarrow B)}{A, \neg B}\,\neg_\Rightarrow$$

$$\frac{\neg(A \wedge B)}{\neg A \quad\quad \neg B}\,\neg_\wedge \qquad \frac{A \vee B}{A \quad\quad B}\,\vee \qquad \frac{A \Rightarrow B}{A \quad\quad \neg B}\,\Rightarrow$$

Example

- prove $(B \vee A) \Rightarrow (A \vee B)$

$$\neg((B \vee A) \Rightarrow (A \vee B))$$

- tableau as a tree
- choice for rule application
- proof iff each branch is closed
- notation $F_1, \cdots, F_n \hookrightarrow \odot$

Example

- prove $(B \vee A) \Rightarrow (A \vee B)$

$$\frac{\neg((B \vee A) \Rightarrow (A \vee B))}{B \vee A, \neg(A \vee B)} \neg\Rightarrow$$

- tableau as a tree
- choice for rule application
- proof iff each branch is closed
- notation $F_1, \cdots, F_n \hookrightarrow \odot$

Example

- prove $(B \vee A) \Rightarrow (A \vee B)$

$$\dfrac{\neg((B \vee A) \Rightarrow (A \vee B))}{\dfrac{B \vee A, \neg(A \vee B)}{B \qquad\qquad A}\ \vee}\ \neg\Rightarrow$$

- tableau as a tree
- choice for rule application
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Example

- prove $(B \vee A) \Rightarrow (A \vee B)$

$$\dfrac{\neg((B \vee A) \Rightarrow (A \vee B))}{\dfrac{B \vee A, \color{red}{\neg(A \vee B)}}{\dfrac{B}{\neg A, \neg B}\ \color{red}{\neg\vee} \qquad A}\ \vee}\ \neg\Rightarrow$$

- tableau as a tree
- choice for rule application
- proof iff each branch is closed
- notation $F_1, \cdots, F_n \hookrightarrow \odot$

Example

- prove $(B \vee A) \Rightarrow (A \vee B)$

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- tableau as a tree
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- proof iff each branch is closed
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Example

- prove $(B \vee A) \Rightarrow (A \vee B)$

$$\dfrac{\neg((B \vee A) \Rightarrow (A \vee B))}{\dfrac{B \vee A, \neg(A \vee B)}{\dfrac{\dfrac{B}{\neg A, \neg B}\,\neg\vee}{\odot} \quad \dfrac{A}{\neg A, \neg B}\,\neg\vee}\,\vee}\,\neg\Rightarrow$$

- tableau as a tree
- choice for rule application
- proof iff each branch is closed
- notation $F_1, \cdots, F_n \hookrightarrow \odot$

Example

- prove $(B \lor A) \Rightarrow (A \lor B)$

$$
\cfrac{\neg((B \lor A) \Rightarrow (A \lor B))}{\cfrac{B \lor A, \neg(A \lor B)}{\cfrac{\cfrac{B}{\neg A, \neg B}\,\neg\lor}{\odot} \qquad \cfrac{\cfrac{A}{\neg A, \neg B}\,\neg\lor}{\odot}}\,\lor}\,\neg\Rightarrow
$$

- tableau as a tree
- choice for rule application
- proof iff each branch is closed
- notation $F_1, \cdots, F_n \hookrightarrow \odot$

Soundness and Completeness for Tableaux

Soundness

If $F_1, \cdots, F_n \hookrightarrow \odot$, then $F_1 \wedge \cdots \wedge F_n$ is unsatisfiable.

- no model of $F_1, \cdots, F_n$
- induction on the tableau proof and case analysis.
- refutation: $\neg F$ unsatisfiable $\rightsquigarrow$ for all interpretations, $[\![F]\!] = 1$

Completeness

If a tableau $F_1, \cdots, F_n$ cannot be closed, then $F_1 \wedge \cdots \wedge F_n$ is satisfiable.

- another view of tableaux rules:
  - ⋆ exhaustively searching for a countermodel
  - ⋆ if for all interpretations, $[\![F]\!] = 1$, then search finds no consistent countermodel on input $\neg F \rightsquigarrow$ closable tableau.

Countermodel from Exhaustion

- try to prove $A \Rightarrow (A \wedge B)$

$$\cfrac{\neg(A \Rightarrow (A \wedge B))}{\cfrac{A, \neg(A \wedge B)}{\cfrac{\neg A}{\odot} \qquad \neg B}}$$

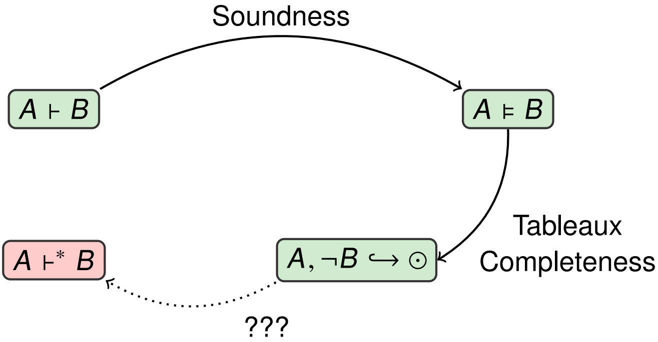
- right branch *open* and *complete*

Complete Branch

A branch of a tableau is complete if all applicable rules have been applied.

- need to construct an exhaustive proof-search algorithm
  - ⋆ collect litterals (plain and negated atoms), A and $\neg B$,
  - ⋆ assign the truth values accordingly, $[\![A]\!] = 1$ and $[\![B]\!] = 0$,
  - ⋆ yields $[\![\neg(A \Rightarrow (A \wedge B))]\!] = 1$ (falsifies $A \Rightarrow (A \wedge B)$).

Completeness and Cut Admissilibity



Sequent Calculus

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \text{axiom} \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ \text{cut}$$

$$\frac{\Gamma A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee_R \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ \Rightarrow_R$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge_R \qquad \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_L \qquad \frac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ \Rightarrow_L$$

- upside down tableaux (two-sided and context duplication)
  - $\neg_\Box$ tableau rule $\sim$ $\Box_R$ rule
  - $\Box$ tableau rule $\sim$ $\Box_L$ rule
  - no cut in the translation

$$\cfrac{\neg((B \vee A) \Rightarrow (A \vee B))}{\cfrac{B \vee A, \neg(A \vee B)}{\cfrac{B}{\cfrac{\neg A, \neg B}{\odot}}\ \neg_\vee \qquad \cfrac{A}{\cfrac{\neg A, \neg B}{\odot}}\ \neg_\vee}\ \vee}\ \neg_\Rightarrow \quad \rightsquigarrow \quad \cfrac{\cfrac{\vee_R\ \cfrac{\overline{B \vdash A, B}}{B \vdash A \vee B} \qquad \vee_R\ \cfrac{\overline{A \vdash A, B}}{A \vdash A \vee B}}{B \vee A \vdash A \vee B}\ \vee_L}{\vdash (B \vee A) \Rightarrow (A \vee B)}\ \Rightarrow_R$$

Tableaux $\rightsquigarrow$ Sequent Calculus

2. Extensions to Other Logics

Switching to First-Order

- variables, terms and quantifiers

$$\forall x(P(x) \Rightarrow P(s(0)))$$

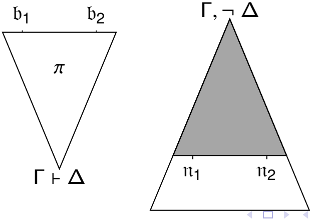
- first-order tableaux, first-order sequent calculus
- cut admissibility by the previous method
- but the complete exhaustive proof-search is highly inefficient
  - ★ enumerates all the terms of the language $t_0, t_1, \cdots$
  - ★ complete branch with $\forall x F$ must have $F[t_0/x], F[t_1/x], \cdots$
  - ★ some sweat to keep proof-search fair

Efficiency in First-Order Tableaux

- unefficient naive enumeration, what if $F[t_{2017}/x]$ right choice?
- do not know: wait to instantiate!
- free variable tableaux

$$
\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x(D(x) \Rightarrow \forall y D(y)))}{\neg(D(X) \Rightarrow \forall y D(y))}\,\neg\exists}{D(X), \neg\forall y D(y)}\,\neg\Rightarrow}{\neg D(c)}\,\neg\forall}{\odot\ \{X \approx c\}}\,\odot
$$

- Exponential speedups, connection lost with sequent calculus
  - ★ freshness condition *globally* ensured, not *locally*
  - ★ re-expand, double inverted induction, duplication



Switching to Deduction Modulo Theory

Rewrite Rule

A term (resp. proposition) rewrite rule is a pair of terms (resp. formulæ) $l \to r$, where $\mathcal{FV}(l) \subseteq \mathcal{FV}(r)$ and, in the propositiona case, l is atomic.

Examples:

- term rewrite rule:

$$A \cup \emptyset \to A$$

- proposition rewrite rule:

$$A \subseteq B \to \forall x \; x \in A \Rightarrow x \in B$$

Conversion modulo a Rewrite System

We consider the congruence $\equiv$ generated by a set of proposition rewrite rules $\mathcal{R}$ and a set of term rewrite rules $\mathcal{E}$ (often implicit)

Example:

$$A \cup \emptyset \subseteq A \quad \equiv \quad \forall x \; x \in A \Rightarrow x \in A$$

(Classical) Sequent Calculus modulo

We add two conversion rules:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash B, \Delta}\ \text{conv}_R, [A \equiv B] \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, B \vdash \Delta}\ \text{conv}_L, [A \equiv B]$$

Or embed conversions modulo $\mathcal{RE}$ directly inside the rules (next slide).

(Classical) Sequent Calculus

$$\frac{}{A \vdash A}\ \text{ax} \qquad\qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ \text{cut}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_L \qquad\qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_L \qquad\qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee_R$$

$$\frac{\Gamma, B \vdash \Delta \qquad \Gamma \vdash A, \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ \Rightarrow_L \qquad\qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ \Rightarrow_R$$

(Classical) Sequent Calculus Modulo

$$\frac{}{A \vdash B}\ \text{ax},\ [A \equiv B] \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma \vdash \Delta}\ \text{cut},\ [A \equiv B]$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, C \vdash \Delta}\ \wedge_L,\ [C \equiv A \wedge B] \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash C, \Delta}\ \wedge_R,\ [C \equiv A \wedge B]$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, C \vdash \Delta}\ \vee_L,\ [C \equiv A \vee B] \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash C, \Delta}\ \vee_R,\ [C \equiv A \vee B]$$

$$\frac{\Gamma, B \vdash \Delta \qquad \Gamma \vdash A, \Delta}{\Gamma, C \vdash \Delta}\ \Rightarrow_L,\ [C \equiv A \Rightarrow B] \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash C, \Delta}\ \Rightarrow_R,\ [C \equiv A \Rightarrow B]$$

Proof of $A \subseteq A$ with and without DM

- without:

$$
\dfrac{\dfrac{\dfrac{\overline{A \subseteq A \Rightarrow [\cdots], x \in A \vdash x \in A, A \subseteq A}}{A \subseteq A \Rightarrow [\cdots] \vdash x \in A \Rightarrow x \in A, A \subseteq A}}{A \subseteq A \Rightarrow [\cdots] \vdash \forall x(x \in A \Rightarrow x \in A), A \subseteq A} \quad \overline{A \subseteq A \Rightarrow [\cdots], A \subseteq A \vdash A \subseteq A}}{\dfrac{A \subseteq A \Rightarrow \forall x(x \in A \Rightarrow x \in A), \forall x(x \in A \Rightarrow x \in A) \Rightarrow A \subseteq A \vdash A \subseteq A}{\dfrac{A \subseteq A \Leftrightarrow \forall x(x \in A \Rightarrow x \in A) \vdash A \subseteq A}{\dfrac{\forall Y(A \subseteq Y \Leftrightarrow \forall x(x \in A \Rightarrow x \in Y)) \vdash A \subseteq A}{\forall X \forall Y(X \subseteq Y \Leftrightarrow \forall x(x \in X \Rightarrow x \in Y)) \vdash A \subseteq A}}}}
$$

- with:

$$
\dfrac{\dfrac{\dfrac{\overline{x \in A \vdash x \in A}}{\vdash x \in A \Rightarrow x \in A} \Rightarrow_R}{\vdash \forall x(x \in A \Rightarrow x \in A)} \forall_R}{\vdash A \subseteq A} \text{conv}_R \; [A \subseteq A \equiv \forall x(x \in A \Rightarrow x \in A)]
$$

Proof of $A \subseteq A$ with and without DM

- without:

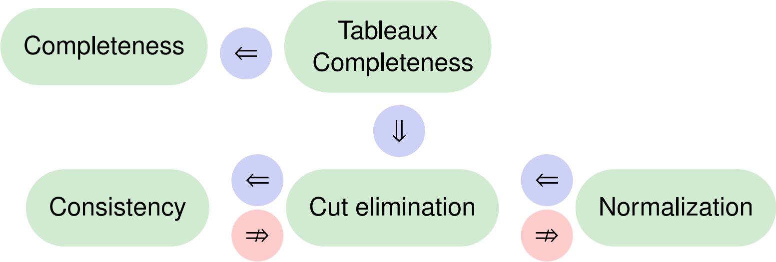
$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \subseteq A \Rightarrow [\cdots], x \in A \vdash x \in A, A \subseteq A}}{A \subseteq A \Rightarrow [\cdots] \vdash x \in A \Rightarrow x \in A, A \subseteq A}}{A \subseteq A \Rightarrow [\cdots] \vdash \forall x (x \in A \Rightarrow x \in A), A \subseteq A} \quad \dfrac{}{A \subseteq A \Rightarrow [\cdots], A \subseteq A \vdash A \subseteq A}}{\dfrac{A \subseteq A \Rightarrow \forall x (x \in A \Rightarrow x \in A), \forall x (x \in A \Rightarrow x \in A) \Rightarrow A \subseteq A \vdash A \subseteq A}{\dfrac{A \subseteq A \Leftrightarrow \forall x (x \in A \Rightarrow x \in A) \vdash A \subseteq A}{\dfrac{\forall Y (A \subseteq Y \Leftrightarrow \forall x (x \in A \Rightarrow x \in Y)) \vdash A \subseteq A}{\forall X \forall Y (X \subseteq Y \Leftrightarrow \forall x (x \in X \Rightarrow x \in Y)) \vdash A \subseteq A}}}}$$

- with:

$$\dfrac{\dfrac{\dfrac{}{x \in A \vdash x \in A}}{\vdash x \in A \Rightarrow x \in A} \Rightarrow_R}{\vdash A \subseteq A} \forall_R$$

Tableaux and Cuts in Deduction Modulo Theory

- beyond first order (axiomless higher-order logic, arithmetic, ...)
- everything depends on $\mathcal{RE}$.
  - consistency ($A \to \neg A$)
  - cut elimination ($A \to (A \Rightarrow A)$)
  - cut admissibility
  - undecidable, even if $\mathcal{RE}$ confluent terminating.



Generic Approach for Tableaux

Nevertheless, genericity:

- as far as possible
  - needs only confluence
  - everything except countermodel construction
- difficulties (besides models)
  - fair and exhausting proof-search design (STEP)
  - interleave quantifier instantiation and rewriting
  - add free-variables
- optimized proof-seach, holes on the complete branch
  - fill the gaps to get a (semi-)valuation
  - not forgetting rewriting

Semantics for Deduction Modulo Theory

- your favorite semantics
- add one constraint

Model of $\mathcal{RE}$

An interpretation $[\![]\!]$ is a model of $\mathcal{RE}$ if for any F, F', such that $F \equiv F'$, we have $[\![F]\!] = [\![F']\!]$.

- straightforward Soundness Theorem

Specific Countermodel Constructions

Completeness of tableaux, hence cut admissibility for

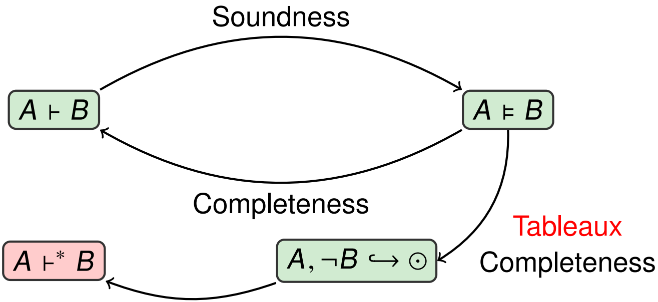
- positive rewrite systems

$$\begin{array}{lcl} \text{even}(0) & \rightarrow & \top \\ \text{even}(S(x)) & \rightarrow & \neg\text{odd}(x) \\ \text{odd}(S(x)) & \rightarrow & \neg\text{even}(x) \end{array}$$

- ordered rewrite systems
- higher-order logic as a rewrite system

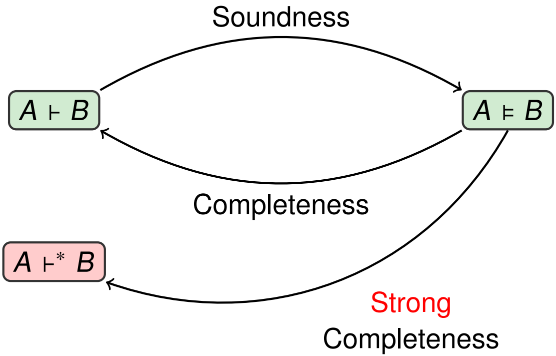
3. Getting Rid of Tableaux

Direct Completeness



- most difficulties in Tableaux Completeness

Direct Completeness



- most difficulties in Tableaux Completeness
- most difficulties in Strong Completeness
  - more flexibility in the semantics
  - 0/1 Boolean algebra (or Kripke structures) imposed by tableaux.

More Flexible Semantics: Algebraic Structures

- propositional intuitionistic logic here (first-order, higher-order possible)
- Heyting algebras
- a universe Ω, operators $\wedge, \vee, \Rightarrow$
- an order $\leq$: Ω is a lattice.
- lowest upper bound (join: $\wedge$), greatest lower bound (meet: $\vee$)

$$a \wedge b \leq a \quad a \wedge b \leq b \quad c \leq a \text{ and } c \leq b \text{ implies } c \leq a \wedge b$$
$$a \leq a \vee b \quad b \leq a \vee b \quad a \leq c \text{ and } b \leq c \text{ implies } a \vee b \leq c$$

- like Boolean algebras (classical case), but
- weak complement (aka implication property):

$$a \wedge b \leq c \text{ iff } a \leq b \Rightarrow c$$

- example: $\mathbb{R}$ and open sets:

$$b \Rightarrow c := \text{ the interior of } b \cup \overline{a}$$

Cut Admissibility: Algebraic Way

Base Elements of the Lindenbaum Algebra

$\lceil A \rceil = \{B \mid A \vdash B \text{ and } B \vdash A\}$

Lidenbaum algebra:

- interpretation of formulas
  - ★ $[\![A]\!] = \lceil A \rceil$ on atoms, then induction
  - ★ $\lceil A \rceil \leq \lceil B \rceil$ iff $A \vdash B$

Fundamental Lemma

For *any* formula A, $[\![A]\!] = \lceil A \rceil$

- what do we have ?

Completeness

if $[\![A]\!] \leq [\![B]\!]$ in all models, then $A \vdash B$.

  - ★ this is the definition of $\leq$ in the Lindenbaum algebra.
- need the cut rule

Cut Admissibility: Algebraic Way

Base Elements of the Lindenbaum Algebra

$\lceil A \rceil = \{B \mid A \vdash B \text{ and } B \vdash A\}$

Cut Admissibility: Algebraic Way

Base Elements of the Context Algebra

$\lceil A \rceil = \{\Gamma \mid \Gamma \vdash A\}$

- $\leq$ is $\subseteq$ and g.l.b. ($\wedge$) and l.u.b. ($\vee$) are "intersection" and "union"
- close Ω by arbitrary intersection:

The Algebra Ω

$$\Omega = \left\{ \bigcap_{C \in \mathcal{C}} \lceil C \rceil \mid \text{ for } \mathcal{C} \text{ set of formulas} \right\}$$

Ω is composed of arbitrary intersections of base elements

- Ω not closed by union
  - there are other ways to compute a least upper bound ...

Cut Admissibility: Algebraic Way

- set the interpretation of the atoms to be: $[\![A]\!] = \lceil A \rceil$

Key Theorem

For *any* formula A, $[\![A]\!] = \lceil A \rceil$.

Cut Admissibility: Algebraic Way

- set the interpretation of the atoms to be: $[\![A]\!] = \lceil A \rceil$

Key Theorem

For *any* formula A, $[\![A]\!] = \lceil A \rceil$.

- what do we have ?

Completeness

if $[\![A]\!] \leq [\![B]\!]$ in all models, then $A \vdash B$.

  - ★ (trivial) $A \in \lceil A \rceil$
  - ★ $\lceil A \rceil = [\![A]\!]$ (Key Theorem)
  - ★ $[\![A]\!] \subseteq [\![B]\!]$ (Hypothesis)
  - ★ $[\![B]\!] = \lceil B \rceil$ (Key Theorem)
  - ★ means $A \vdash B$

Cut Admissibility: Algebraic Way

- set the interpretation of the atoms to be: $[\![A]\!] = \lceil A \rceil$

Key Theorem

For *any* formula A, $[\![A]\!] = \lceil A \rceil$.

- what do we really need ?

Completeness

if $[\![A]\!] \leq [\![B]\!]$ in all models, then $A \vdash B$.

- ★
- ★ $A \in [\![A]\!]$ (Key Theorem)
- ★ $[\![A]\!] \subseteq [\![B]\!]$ (Hypothesis)
- ★ $[\![B]\!] \subseteq \lceil B \rceil$ (Key Theorem)
- ★ means $A \vdash B$

Cut Admissibility: Algebraic Way

- Ω contains arbitrary intersections of base elements.

Base Elements

$\lceil A \rceil = \{\Gamma \mid \Gamma \vdash A\}$

- $\leq$ is $\subseteq$. Gives a lattice. Also a Heyting algebra.
- set the interpretation of the atoms to be: $[\![A]\!] = \lceil A \rceil$

Key Theorem

For *any* formula A, $[\![A]\!] = \lceil A \rceil$.

- what do we have ?

Completeness

if $[\![A]\!] \leq [\![B]\!]$ in all models, then $A \vdash B$.

Proof: $A \in \lceil A \rceil = [\![A]\!] \subseteq [\![B]\!] = \lceil B \rceil$.

Cut Admissibility: Algebraic Way

- Ω contains arbitrary intersections of base elements.

Base Elements

$\lceil A \rceil = \{\Gamma \mid \Gamma \vdash^* A\}$

- $\leq$ is $\subseteq$. Gives a lattice. Also a Heyting algebra ($\Rightarrow$ property difficult)
- set the interpretation of the atoms to be: $[\![A]\!] = \lceil A \rceil$

Key Theorem

For *any* formula A, $A \in [\![A]\!] \subseteq \lceil A \rceil$

- Similarities with Reducibility Candidate-models (Logical Relations)

$$NE \subseteq \mathcal{R}_A \subseteq SN \quad \text{(simplified)}$$

Strong Completeness

if $[\![A]\!] \leq [\![B]\!]$ in all models, then $A \vdash^* B$.

Proof: $A \in \lceil A \rceil = [\![A]\!] \subseteq [\![B]\!] = \lceil B \rceil$.

Cut Admissibility, Second Order: Algebraic Way

- Ω contains arbitrary intersections of base elements.

Base Elements

$\lceil A \rceil = \{\Gamma \mid \Gamma \vdash^* A\}$

- $\leq$ is $\subseteq$. Gives a lattice. Also a Heyting algebra ($\Rightarrow$ property difficult)
- set the interpretation of the atoms to be: $[\![A]\!] = \lceil A \rceil$

Key Theorem

For *any* formula A, $A\sigma \in [\![A]\!]_\phi \subseteq \lceil A\sigma \rceil$,
for any ϕ, σ such that $\sigma(X_i) \in \phi(X_i) \subseteq \lceil \sigma(X_i) \rceil$

- Similarities with Reducibility Candidate-models (Logical Relations)

$$NE \subseteq \mathcal{R}_A \subseteq SN \quad \text{(simplified)}$$
$$[\![A]\!]_\phi \in \mathcal{R}_{A\sigma}, \text{ for any } \phi, \sigma \text{ s.t. } \phi(X_i) \in R_{\sigma(X_i)}$$

Strong Completeness

if $[\![A]\!] \leq [\![B]\!]$ in all models, then $A \vdash^* B$.

Application to Higher-Order Logics

- does not apply directly to higher-order logic
- intensional logic

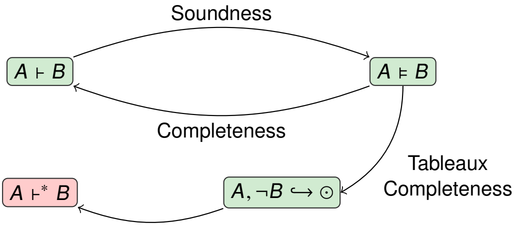
$$P(\top) \not\Leftrightarrow P(\top \wedge \top)$$

- $[\![\top]\!] \neq \top$
- V-complexes [Takahashi], [Prawitz], [Andrews]
- adapted to
  - ★ intuitionnistic case,
  - ★ linear case (phase semantics),
  - ★ the Deduction modulo theory expression of HOL (classical and intuitionnistic).

4. Opening the Box

Inside Constructive Proofs

- Cut admissibility through tableaux, almost constructive
  - ★ rebuild proof from scratch



- Henkin completeness proofs

Computational Content of Algebraic Proofs

- switch to Natural Deduction
- more work existing
  - Normalization by Evaluation
  - all Kripke (-like)
- easier to compare
  - and understand (at least, so we thought)
  - no problem with disjunction in Heyting algebra

What Had to be Done

- from Sequent Calculus to Natural Deduction

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- from Sequent Calculus to Natural Deduction
- notion of cut-free proof

Cut-Free Proofs

A proof is neutral it is an elimination with cut-free premises and neutral principal premiss. A proof is cut-free it is an introduction with cut-free premises.

$$\frac{\Gamma \vdash_{ne} A}{\Gamma \vdash^* A}\ coerce \qquad \frac{A \in \Gamma}{\Gamma \vdash_{ne} A}\ ax$$

$$\frac{\Gamma \vdash^* A \qquad \Gamma \vdash^* B}{\Gamma \vdash^* A \wedge B}\ \wedge_I \qquad \frac{\Gamma \vdash_{ne} A \wedge B}{\Gamma \vdash_{ne} A}\ \wedge_{E_l} \qquad \frac{\Gamma \vdash_{ne} A \wedge B}{\Gamma \vdash_{ne} B}\ \wedge_{E_r}$$

$$\frac{\Gamma \vdash^* A}{\Gamma \vdash^* A \vee B}\ \vee_{I_l} \qquad \frac{\Gamma \vdash^* B}{\Gamma \vdash^* A \vee B}\ \vee_{I_r} \qquad \frac{\Gamma \vdash_{ne} A \vee B \qquad A, \Gamma \vdash^* C \qquad B, \Gamma \vdash^* C}{\Gamma \vdash_{ne} C}\ \vee_E$$

$$\frac{\Gamma, A \vdash^* B}{\Gamma \vdash^* A \Rightarrow B}\ \Rightarrow_I \qquad \frac{\Gamma \vdash_{ne} A \Rightarrow B \qquad \Gamma \vdash^* A}{\Gamma \vdash_{ne} B}\ \Rightarrow_E$$

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- show that constructions are still valid

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  - ★ or universe inconsistency

What had to be Done - 2

- works for first-order logic (probably more)
- formalize in Coq (propositional logic)
- extract the algorithm:
  - ★ limitations of Coq
  - ★ either we face proof-irrelevance
  - ★ or universe inconsistency
- we can at least observe inside Coq
- or have a potentially unsound algorithm

On Examples

- how a ⇒-cut is reduced

$$\dfrac{\dfrac{\dfrac{}{A, A \vdash A}\,ax}{A \vdash A \Rightarrow A}\Rightarrow_I \qquad \dfrac{}{A \vdash A}\,ax}{A \vdash A}\Rightarrow_E \quad \rhd \quad \dfrac{}{A \vdash A}\,ax$$

On Examples

- how a $\vee$-cut is reduced

$$
\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax}{A \vdash A \vee A}\,\vee_{I_l} \quad \dfrac{\dfrac{}{A, {\color{red}A} \vdash {\color{red}A}}\,ax}{A, {\color{red}A} \vdash A \vee A}\,\vee_{I_r} \quad \dfrac{\dfrac{}{A, {\color{red}A} \vdash {\color{red}A}}\,ax}{A, {\color{red}A} \vdash A \vee A}\,\vee_{I_l}}{A \vdash A \vee A}\,\vee_E \quad \rhd \quad \dfrac{\dfrac{}{A \vdash A}\,ax}{A \vdash A \vee A}\,\vee_{I_r}
$$

On Examples

- η-expansion

$$\dfrac{}{A \vee B \vdash A \vee B}\ ax$$

On Examples

- η-expansion

$$\dfrac{\dfrac{}{A \vee B \vdash A \vee B}\,ax \qquad \dfrac{\dfrac{}{A \vee B, A \vdash A}\,ax}{A \vee B, A \vdash A \vee B}\,\vee_{I_l} \qquad \dfrac{\dfrac{}{A \vee B, B \vdash B}\,ax}{A \vee B, B \vdash A \vee B}\,\vee_{I_r}}{A \vee B \vdash A \vee B}\,\vee_E$$

On Examples

- η-expansion, one more step

$$\dfrac{\dfrac{\dfrac{}{A \lor B \vdash A \lor B}\,ax \qquad \dfrac{\dfrac{\dfrac{}{A \lor B, A, A \lor B \vdash A}\,ax}{A \lor B, A, A \lor B \vdash A \lor B}\,\lor_{I_l}}{A \lor B, A \vdash (A \lor B) \Rightarrow (A \lor B)}\,\Rightarrow_I \qquad \dfrac{\dfrac{\dfrac{}{A \lor B, B, A \lor B \vdash B}\,ax}{A \lor B, B, A \lor B \vdash A \lor B}\,\lor_{I_r}}{A \lor B, B \vdash (A \lor B) \Rightarrow (A \lor B)}\,\Rightarrow_I}{A \lor B \vdash (A \lor B) \Rightarrow (A \lor B)}\,\lor_E \qquad \dfrac{}{A \lor B \vdash A \lor B}\,ax}{A \lor B \vdash A \lor B}\,\Rightarrow_E$$

Conclusion

Computational content of algebraic methods:

- still to explore
- commutative cuts

A lot of domains:

- logics with constraints (higher order)
- polarized Deduction Modulo Theory
  - model theory
  - theoretical results
  - tools
  - better Skolem symbols (rewriting)

This is first order, no dependent types

- $\lambda\Pi$-calculus Modulo Theory
- Dedukti