# Semantic Cut Elimination in Intuitionnistic Deduction Modulo

Olivier Hermant, INRIA Futurs, Palaiseau
http ://pauillac.inria.fr/~ohermant

Types 2004
Dec. 15-18 Jouy-en-Josas

## Outline

- The deduction system
- Soundness, Completeness and Cut Elimination
- Two conditions and the proofs
- Putting the two conditions together

## Some Rules of Sequent Calculus

$$\frac{}{\Gamma, P \vdash P}\text{axiom} \qquad \frac{\Gamma, P \vdash R \quad \Gamma \vdash P}{\Gamma \vdash R}\text{cut}$$

$$\frac{\Gamma \vdash P \quad \Gamma Q \vdash R}{\Gamma P \Rightarrow Q \vdash R} \Rightarrow\text{-l} \qquad \frac{\Gamma, P \vdash Q}{\Gamma \vdash P \Rightarrow Q} \Rightarrow\text{-r}$$

$$\frac{\Gamma, \{t/x\}P \vdash R}{\Gamma, \forall x P \vdash R}\forall\text{-l} \qquad \frac{\Gamma \vdash \{c/x\}P}{\Gamma \vdash \forall x P}\forall^*\text{-r}$$

We want to add rewrite rules on terms and on propositions :

$$\begin{aligned} x * y = 0 &\quad\longrightarrow\quad (x = 0) \vee (y = 0) \\ (x + y) + z &\quad\longrightarrow\quad x + (y + z) \\ x * 0 &\quad\longrightarrow\quad 0 \end{aligned}$$

$\mathcal{R}$ is a set of Rewrite Rules

Some Rules of Sequent Calculus Modulo

$$\frac{}{\Gamma, P \vdash_{\mathcal{R}} Q}\text{axiom},\ P \equiv Q \qquad \frac{\Gamma, P \vdash_{\mathcal{R}} R \quad \Gamma \vdash_{\mathcal{R}} Q}{\Gamma \vdash_{\mathcal{R}} R}\text{cut},\ P \equiv Q$$

$$\frac{\Gamma \vdash_{\mathcal{R}} P \quad \Gamma, Q \vdash_{\mathcal{R}} R}{\Gamma S \vdash_{\mathcal{R}} R} \wedge\text{-l},\ S \equiv P \Rightarrow Q \qquad \frac{\Gamma, P \vdash_{\mathcal{R}} Q}{\Gamma \vdash_{\mathcal{R}} S} \Rightarrow\text{-r},\ S \equiv P \Rightarrow Q$$

$$\frac{\Gamma, \{t/x\}P \vdash_{\mathcal{R}} R}{\Gamma, Q \vdash_{\mathcal{R}} R}\forall\text{-l},\ Q \equiv \forall x P \qquad \frac{\Gamma \vdash_{\mathcal{R}} \{c/x\}P}{\Gamma \vdash_{\mathcal{R}} Q}\forall^*\text{-r},\ Q \equiv \forall x P$$

In the general case, cut elimination doesn't hold (even in confluent terminating cases) :

$$A \quad \longrightarrow \quad B \wedge \neg A$$

$\Rightarrow$ We need some (general) conditions on the Rewrite System to ensure cut-elimination.

## Classical case

**Theorem[Soundness]** : If $\Gamma \vdash_{\mathcal{R}} \Delta$ (with possible cuts) then $\Gamma \models \Delta$.

**Theorem[Completeness]** : If $\mathcal{T}$ is a cut free-consistent theory, it has a model

**Corollary[Cut elimination]** : If $\Gamma \vdash_{\mathcal{R}} \Delta$ then $\Gamma \models \Delta$ hence $\Gamma \vdash_{\mathcal{R}}^{cf} \Delta$.

In the intuitionnistic case, we need other definitions than the usual ones :

| | | |
|---|---|---|
| $A$**-consistency** | : | $\Gamma \not\vdash^{cf}_{\mathcal{R}} A$ |
| $A$**-completeness** | : | $P \in \Gamma$ or $\Gamma, P \vdash^{cf}_{\mathcal{R}} A$ |
| $A$**-Henkin witnesses** | : | $\Gamma, \exists x P \not\vdash^{cf}_{\mathcal{R}} A \Rightarrow \{c/x\}P \in \Gamma$ |

**Theorem[Soundness]** : If $\Gamma \vdash_{\mathcal{R}} P$ (with possible cuts) then $\Gamma \models P$.

**Theorem[Completeness]** : If $\mathcal{T}$ is a $P$-cut free-consistent theory, it has a model that is not a model of $P$.

**Corollary[Cut elimination]** : If $\Gamma \vdash_{\mathcal{R}} P$ then $\Gamma \vdash^{cf}_{\mathcal{R}} P$.

Proof : if $\Gamma \vdash_{\mathcal{R}} P$, by soundness, we have $\Gamma \models P$. By completeness theorem, this means that $\Gamma$ is $P$-cut free-inconsistent, i.e. $\Gamma \vdash^{cf}_{\mathcal{R}} P$.

Our notion of model : Kripke Model, extended to deduction modulo with the condition :

$$\text{if } P \equiv Q \text{ then } \alpha \Vdash P \Leftrightarrow \alpha \Vdash Q$$

A Kripke Structure :

- a partially ordered set of "worlds" $K, \leq$
- a domain $D$ for each $\alpha \in K$, $D(\alpha)$. $D$ is monotone wrt $\leq$.
- a forcing relation $\Vdash$ defined by induction on the propositions. e.g.

$$\alpha \Vdash P \Rightarrow Q \;\text{ iff }\; \forall \beta \geq \alpha \;\; \beta \Vdash P \Rightarrow \beta \Vdash Q$$

## Completion of an $A$-consistent theory $\mathcal{T}$

Set $\Gamma_0 = \mathcal{T}$, enumerate all the propositions of the language extended with new fresh constants :

$$P_0, ..., P_n, ...$$

At each step, check if $\Gamma_n, P_n \vdash_{\mathcal{R}}^{cf} A$ or not, and define

– $\Gamma_{n+1} = \Gamma_n$ if yes

– $\Gamma_{n+1} = \Gamma_n \cup \{P_n\}$ if no

Add a Henkin witness if $P_n$ is an existential formula.

Take $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n$.

$\Gamma$ is $A$-complete, $A$-consistent, admits $A$-Henkin witnesses.

All of this is valid under the only hypothesis of confluence of $\mathcal{R}$

# First condition

We will consider a rewrite system that is :

- confluent
- terminating
- compatible with a well-founded order $\succ$ having the subformula property.

E.g. the rule $A[x, 0] \longrightarrow B[x] \Rightarrow C$ is compatible with such an order.

We prove the completeness theorem : given a $A$-consistent theory $\mathcal{T}$ expressed in a language $\mathcal{L}_0$, we construct a Kripke Structure and find a node $\alpha \Vdash \mathcal{T}$ and $\alpha \nVdash A$

# Model Construction

Consider a denumerable family of set of new constants $C_n$.
Define the languages $L_0, ..., L_n$ such that $L_{n+1} = L_n \cup C_n$,

- $K = \{\Gamma\}$, s.t. for some proposition $A$, $\Gamma$ is an $A$-consistent, $A$-complete theory of some $L_i$, admitting $A$-Henkin witnesses.
- $\leq$ is the inclusion.
- $D(\Gamma) = clos(L_i)$
- if $A$ is a normal atom, $\Gamma \Vdash A$ iff $A \in \Gamma$.
- if $A$ is a non-normal atom, set $\Gamma \Vdash A$ iff $\Gamma \Vdash A \downarrow$.
- define as usual the interpretation of non atomic propositions.

Definition is well-founded, thanks to $\succ$. This is a Kripke Structure for $\mathcal{R}$, and $\Gamma \Vdash \mathcal{T}$, $\Gamma \not\Vdash A$

## Application : Quantifier-free rewrite systems

We consider only rules $A \to Q$ where $Q$ doesn't contain quantifiers. We need confluence and termination of the set of rules.

The pair $< \#_{\forall,\exists}, \#_{\wedge,\vee,\Rightarrow,\neg} >$ is a well-founded order on normal terms.

Extend it on propositions : $A \succ B$ if

- $A{\downarrow} \succ B{\downarrow}$
- $A{\downarrow} = B{\downarrow}$ and $A \to^{+} B$

# Second condition

We consider a positive rewrite system $\mathcal{R}$ :
in a rewrite rule $A \rightarrow P$ atoms of $P$ occurs positively. For example :

$$A \longrightarrow \forall x A$$
$$A \longrightarrow (\neg B) \Rightarrow C$$

# Model construction

As before, we define a family of languages.

- $K = \{\Gamma\}$, $\Gamma$ is an $A$-consistent, $A$-complete theory of some $L_i$, admitting $A$-Henkin witnesses, ordered by $\subset$
- $D(\Gamma) = clos(L_i)$
- in a world $\Gamma$, we define the truth value of **all** atoms, and extend it on all propositions.
  - If $B \in \Gamma$ is atomic, we let $\Gamma \Vdash B$
  - if $\Gamma \vdash_{\mathcal{R}}^{cf} B$ and $\Gamma, B \vdash_{\mathcal{R}}^{cf} A$ we let $\Gamma \Vdash B$.
  - else, $\Gamma \nvdash_{\mathcal{R}}^{cf} B$, we let $\Gamma \nVdash B$.

It is a Kripke Structure, $\Gamma \Vdash \Gamma$ and $\Gamma \nVdash A$.

But is it a model of the rewrite rule ? The key lemma :

**Lemma 1**

$$\begin{array}{cc} \Gamma, P^+ \vdash_{\mathcal{R}}^{cf} A & \Gamma, Q^- \vdash_{\mathcal{R}}^{cf} A \\ \Gamma \vdash_{\mathcal{R}}^{cf} P^+ & \Gamma \vdash_{\mathcal{R}}^{cf} Q^- \end{array}$$

*implies*

$$\begin{array}{cc} \Gamma \Vdash P^+ & \Gamma \nVdash Q^- \end{array}$$

## Two conditions together

- R $= \mathcal{R}_{\succ} \cup \mathcal{R}_{+}$
- where $\mathcal{R}_{\succ}$ is compatible with a wfo
- and $\mathcal{R}_{+}$ a positive rewrite system such that
- for any atomic proposition $A$ normal for $\mathcal{R}_{\succ}$, any $P$, if $A \equiv_{\mathcal{R}_{+}} P$ then any instance of any atom of $P$ is normal for $\mathcal{R}_{\succ}$.

Example :

$$
\begin{array}{rcl}
A \rightarrow & (\forall x B) \wedge C & \text{compatible with an order} \\
B(0) \rightarrow & \forall x B & \text{a positive rewrite rule}
\end{array}
$$

# Model Construction

We define the Kripke Structure as usual except of the forcing relation :

- if $A$ is a normal atom for $\mathcal{R}_{\succ}$, $\Gamma \Vdash A$ iff $\Gamma \vdash^{cf}_{\mathcal{R}} A$.
- if $A$ is a non-normal atom for $\mathcal{R}_{\succ}$, set $\Gamma \Vdash A$ if $\Gamma \Vdash A\downarrow_{\succ}$.
- in the non-atomic case, set the forcing relation according to the Kripke Structure definition

- We have that $\Gamma \Vdash \Gamma$ and $\Gamma \nVdash A$
- We get a model for $\mathcal{R}_{\succ}$ : proof as in the order case.
- We prove as in the positive case, the lemma :

  **Lemma 2**

$$\begin{array}{rcl} \Gamma, P^{+} \vdash^{cf}_{\mathcal{R}} A & & \Gamma, Q^{-} \vdash^{cf}_{\mathcal{R}} A \\ \Gamma \vdash^{cf}_{\mathcal{R}} P^{+} & & \Gamma \vdash^{cf}_{\mathcal{R}} Q^{-} \\ & \textit{implies} & \\ \Gamma \Vdash P^{+} & & \Gamma \nVdash Q^{-} \end{array}$$

  and the Kripke Structures yields a model for $\mathcal{R}_{+}$ too. $\Box$

## Conclusion and Perspectives

- link with strong normalization and pre-model construction ([Dowek,Werner])
  - normalization is NOT cut elimination
  - however, how to transform pre-models into Kripke Structure ? ([Dowek],[Coquand, Gallier])
- extend this result to the intuitionnistic first-order expression of HOL.

Deduction rules of the Intuitionnistic Sequent Calculus Modulo

$$\frac{}{\Gamma, P \vdash_{\mathcal{R}} Q}\text{axiom if } P \equiv Q \qquad \frac{\Gamma, P \vdash_{\mathcal{R}} Q \quad \Gamma \vdash_{\mathcal{R}} R}{\Gamma \vdash_{\mathcal{R}} Q}\text{cut if } P \equiv R$$

$$\frac{\Gamma, P, R \vdash_{\mathcal{R}} Q}{\Gamma, P \vdash_{\mathcal{R}} Q}\text{contr-l if } P \equiv R \qquad \frac{}{\Gamma, P \vdash_{\mathcal{R}} Q}\bot\text{-l if } P \equiv \bot$$

$$\frac{\Gamma \vdash_{\mathcal{R}} Q}{\Gamma, P \vdash_{\mathcal{R}} Q}\text{weak-l} \qquad \frac{\Gamma \vdash_{\mathcal{R}}}{\Gamma \vdash_{\mathcal{R}} P}\text{weak-r}$$

$$\frac{\Gamma, P, Q \vdash_{\mathcal{R}} R}{\Gamma, S \vdash_{\mathcal{R}} R}\wedge\text{-l if } P \wedge Q \equiv S \qquad \frac{\Gamma \vdash_{\mathcal{R}} P \quad \Gamma \vdash_{\mathcal{R}} Q}{\Gamma \vdash_{\mathcal{R}} R}\wedge\text{-r if } P \wedge Q \equiv R$$

$$\frac{\Gamma, P \vdash_{\mathcal{R}} R \quad \Gamma, Q \vdash_{\mathcal{R}} R}{\Gamma, S \vdash_{\mathcal{R}} R}\vee\text{-l if } P \vee Q \equiv S \qquad \frac{\Gamma \vdash_{\mathcal{R}} P}{\Gamma \vdash_{\mathcal{R}} S}\vee\text{-r if } P \vee Q \equiv R \qquad \frac{\Gamma \vdash_{\mathcal{R}} Q}{\Gamma \vdash_{\mathcal{R}} R}\vee\text{-r if } P \vee Q \equiv R$$

$$\frac{\Gamma \vdash_{\mathcal{R}} P \quad \Gamma, Q \vdash_{\mathcal{R}} R}{\Gamma, S \vdash_{\mathcal{R}} R}\Rightarrow\text{-l if } P \Rightarrow Q \equiv S \qquad \frac{\Gamma, P \vdash_{\mathcal{R}} Q}{\Gamma \vdash_{\mathcal{R}} S}\Rightarrow\text{-r if } P \Rightarrow Q \equiv S$$

$$\frac{\Gamma \vdash_{\mathcal{R}} P}{\Gamma, Q \vdash_{\mathcal{R}}}\neg\text{-l if } \neg P \equiv Q \qquad \frac{\Gamma, P \vdash_{\mathcal{R}}}{\Gamma \vdash_{\mathcal{R}} Q}\neg\text{-r if } \neg P \equiv Q$$

$$\frac{\Gamma, \{t/x\}P \vdash_{\mathcal{R}} Q}{\Gamma, R \vdash_{\mathcal{R}} Q}\forall\text{-l if } \forall x P \equiv R \qquad \frac{\Gamma \vdash_{\mathcal{R}} \{c/x\}P}{\Gamma \vdash_{\mathcal{R}} Q}\forall^*\text{-r, if } \forall x P \equiv Q$$

$$\frac{\Gamma, \{c/x\}P \vdash_{\mathcal{R}} Q}{\Gamma, R \vdash_{\mathcal{R}} Q}\exists^*\text{-l, if } \exists x P \equiv R \qquad \frac{\Gamma \vdash_{\mathcal{R}} \{t/x\}P}{\Gamma \vdash_{\mathcal{R}} Q}\exists\text{-r if } \exists x P \equiv Q$$